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Spectrum generating algebras for the classical Kepler problem

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Abstract
It is shown that the *so*(4, 2) spectrum generating algebra for the classical Kepler problem for non-zero energies can be obtained from the generators of the spacetime conformal group *SO*(4, 2). This is achieved by exploiting the equivalence of Kepler motion and null geodesic motion in conformally flat Einstein static spacetimes. We show that it is the existence of a time-dependent representation of the *so*(4, 2) spectrum generating algebra for null geodesic motion in the Einstein static spacetimes (originating from the *so*(4, 2) algebra of first integrals) which determines the corresponding spectrum generating algebra structure in the classical Kepler problem. Further, for the zero energy state, it is shown that only the *iso*(3) invariance subalgebra has a direct physical significance.

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1. Introduction

Consider an arbitrary Hamiltonian system with Hamiltonian function $H(x, y)$, where x and y are conjugate coordinates and momenta respectively, and let λ be the time parameter. The first integrals of motion $C(x, y; \lambda)$ satisfy

$$\frac{\partial C}{\partial \lambda} + \{H, C\} = 0 \tag{1}$$

and form a Lie algebra. The time-independent first integrals of motion $C(x, y)$ satisfy the Poisson bracket relation

$$\{H, C\} = 0. \tag{2}$$

The set of time-independent first integrals form a subalgebra and constitute a representation of the *invariance algebra* or *symmetry algebra* of the Hamiltonian system [1]. Equation (1) is the classical version of the following statement: if $\psi(x; \lambda)$ is any solution of the time-dependent Schrödinger equation

$$\left[\mathrm{i}\frac{\partial}{\partial \lambda} - H(x, y)\right]\psi(x; \lambda) = 0 \tag{3}$$

then $C(x, y; \lambda)\psi(x; \lambda)$ is also a solution of the same equation if C satisfies

$$\mathrm{i}\frac{\partial C}{\partial \lambda} - \{H, C\} = 0. \tag{4}$$

If $\partial C/\partial\lambda \neq 0$ then $C\psi$ is a linear combination of eigenstates of H with different energies, that is C generates the spectrum of H. Thus the Lie algebra of (time-dependent and time-independent) first integrals is said to form a *time-dependent representation* (TDR) of the *spectrum generating algebra* of the Hamiltonian system [2–5]. We shall only be concerned with finite-dimensional spectrum generating algebras. The spectrum generating algebra is also known as the *non-invariance* or *dynamical* algebra. The elements of the spectrum generating algebra are the infinitesimal generators of the spectrum generating group of the Hamiltonian system, which is the group of transformations mapping orbits into orbits [5]. The subalgebra with $\partial C/\partial\lambda = 0$ is the invariance algebra and under quantization, this invariance leads to the degeneracy of the energy levels of the dynamical system and the associated operators commute with the Hamiltonian [2]. If $C(x, y; \lambda)$ is a first integral then $\partial C/\partial\lambda$ is a first integral and (1) implies that $\{H, C\}$ is also a first integral. It follows that, given a Lie algebra of first integrals, either H is an element of that Lie algebra or the Lie algebra can be supplemented by H to generate a larger Lie algebra. In both cases, we have that the Hamiltonian H *maps the Lie algebra into itself* by the Poisson bracket operation, i.e. if we label the basis first integrals $C_J(x, y; \lambda)$, $J = 1, \ldots, r$, then

$$\{H, C_I\} = D_I^J C_J \tag{5}$$

for some structure constants D_I^J, $I = 1, \ldots, r-1$, and $C_r = H$. Thus, (1) and (5) can be thought of as equivalent definitions. Further, if we consider the stationary system $\lambda = 0$, then the quantities $C_I(x, y; \lambda = 0)$ form a *time-independent representation* (TIR) of the algebra. The spectrum generating group is a non-compact group, an irreducible representation space of which contains *all* the states of the system and the non-compact generators of which are associated with the energy operator [6–18].

In this paper, we construct the spectrum generating algebra for the classical Kepler problem, i.e. the Hamiltonian system with Hamiltonian

$$H = \frac{|y|^2}{2} - \frac{\alpha}{|x|} \tag{6}$$

by exploiting the equivalence between Kepler motion and geodesic motion on a related manifold. From the introductory remarks above, we can see that the problem reduces to that of finding the algebra of (time-dependent and time-independent) first integrals of Kepler motion, and this approach has a number of advantages over previous methods. In other words, we determine a TDR of the spectrum generating algebra.

First we review known results. The three-dimensional classical Kepler problem is known to admit three further first integrals of motion, in addition to the components of angular momentum [19–24]. The bound states in the quantized Kepler problem have $SO(4)$ symmetry, which was used to solve for the energy spectrum, and explain the degeneracy, of the hydrogen atom [25–28]. In fact, the classical Kepler problem has $SO(4)$, $ISO(3)$ and $SO(3, 1)$ symmetry for negative, zero and positive energy orbits respectively [1]. Thus there is a one-to-one correspondence between the symmetry algebras for motion in the classical Kepler problem and geodesic motion on three-dimensional spaces of constant curvature, for each energy surface (i.e. the three-sphere with positive curvature, Euclidean three-space with zero curvature and the three-hyperboloid with negative curvature). This is due to the equivalence of Hamiltonian flows on the phase space [29–35]. Bacry [36] demonstrated that, for bound states in the classical Kepler problem, the generators of the $SO(4)$ invariance group can be supplemented to give a TIR of an $SO(4, 1)$ non-invariance group. Bander and Itzykson show that the

$SO(4, 1)$ group connects various energy levels in the hydrogen atom [37] and also in the case of scattering states [38]. Sudarshan and Mukunda [39] consider the action of these non-invariance groups from both a quantum mechanical and classical point of view. This new group can be further extended to generate the full spectrum generating group $SO(4, 2)$ [40–43] and has immediate application to the group dynamics of the hydrogen atom [44–48]. It should be noted that the $SO(2, 1)$ subgroup is sufficient to generate an *energy* spectrum depending only on the principal quantum number but does not give any information on the degeneracies. $SO(4, 2)$ contains all the operators necessary to allow transitions between arbitrary negative energy (or positive energy) states [47–49]. The case of zero energy has largely been avoided in the literature, with the exception of Barut [49] (see also Lindblad and Nagel [50]) who shows that the $SO(2, 1)$ generators for the non-zero energy states do not provide an energy spectrum for the case $H = E = 0$.

The appearance of the Lie group $SO(4, 2)$ requires explanation. The Lie group $SO(4, 2)$ arises in a number of physical systems. This Lie group is defined to be the isotropy group in $\mathbb{R}^{4,2}$ and it also arises as the conformal symmetry group of Minkowski spacetime and hence all conformally flat spacetimes [51–55]. Barut and Bornzin [56] consider various ways to unify the spacetime conformal group with the dynamical group but state in the introduction to their paper that a conclusive and final answer is yet to be found. In [56], various methods are devised for unifying the group structures, using both a six-dimensional and a four-dimensional approach, i.e. physical processes are considered to take place in a six- or four-dimensional 'Minkowski space' with a suitable projection to obtain the familiar dynamical groups. Souriau [32] refers to the space of non-zero covectors of S^3, denoted by T^+S^3, as the *Kepler manifold* and shows that T^+S^3 is a minimal co-adjoint orbit of $SO(4, 2)$, see also [57–62]. Baumgarte [63] derives the non-invariance Lie algebra $so(4, 2)$ and then constructs the KS-transformation, which maps the three-dimensional Kepler motion into the four-dimensional oscillator (see also [64]). Kummer [65] relates Moser's regularization procedure to the KS-regularization for the three-dimensional classical Kepler problem (for non-zero energies) and discusses the role of $SO(4, 2)$ and $SU(2, 2)$ in these schemes. Iwai [66] and Mladenov [67] associate the four-dimensional harmonic oscillator with the three-dimensional Kepler problem and MIC-Kepler problem respectively. Guillemin and Sternberg [62] show that the Kepler motion can be enlarged to geodesic flow on a curved Lorentzian five-dimensional manifold. In their work, the mass parameter is directly related to a conjugate momentum coordinate in the cotangent bundle. Cordani [54] (see also [55]) derives the conformal symmetry generators in conformally flat spacetimes (in an inverted coordinate system) from the isotropy group generators in $\mathbb{R}^{4,2}$. Cordani then provides a (energy-dependent) canonical transformation on the eight-dimensional phase space giving the equations of motion for the classical Kepler problem (which applies in the case of non-zero energies only) and then obtains a TIR of the $so(4, 2)$ algebras for non-zero energies by taking an appropriate hypersurface (time equal to zero). Cariñena *et al* [68] investigate the conformal geometry of both the Kepler orbit configuration space and momentum space. They determine the action of an $SO(3, 2)$ Lie group in the configuration space and then show the dynamical role of another realization of the $SO(3, 2)$ Lie group and these group actions are shown not to be equivalent.

In [69] it was shown how the methods of [29–31] could be unified in a *continuous* way (in particular there is no energy rescaling of the type used in, for example, [29, 30, 54, 65]). In this paper we extend the method of [69], exploiting the continuity wherever possible, to generate the time-dependent first integrals of Kepler motion. The Einstein static spacetimes are foliated by three-dimensional spaces of constant curvature k parametrized by a time coordinate. Thus it is natural to write the equations of geodesic motion on such a three-dimensional space in terms of geodesic motion on such a static four-dimensional spacetime manifold. These spacetimes are conformally flat and so admit 15 conformal Killing vector fields with Lie algebra $so(4, 2)$ [70]. In the case of null geodesics only, all 15 conformal Killing vectors yield first integrals of motion and the corresponding Poisson bracket Lie algebra is also $so(4, 2)$. Here we extend the procedure in [69] so that these time-dependent first integrals become time-dependent first integrals of motion for the classical Kepler problem. This extension is based upon the following. Given the flow

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\partial}{\partial\lambda} + \mathcal{F}\hat{\mathbf{X}}^3_H \tag{7}$$

it can be reparametrized by $\mathrm{d}\tau = \mathcal{F}\,\mathrm{d}\lambda$ (τ is the Kepler time, λ is the eccentric anomaly) to give

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \mathcal{F}\frac{\mathrm{d}}{\mathrm{d}\tau} = \mathcal{F}\left[\frac{\partial}{\partial\tau} + \hat{\mathbf{X}}^3_H\right] \tag{8}$$

which guarantees that the first integrals of motion for the former system are also first integrals of motion for the latter [71]. In general such a parameter transformation is non-canonical. However, for the case of non-zero energies $H = E \neq 0$ this *can* be made to correspond to a canonical transformation and so preserves the Poisson bracket structures. Cordani [54] explains the appearance of a TIR of the $so(4, 2)$ algebra satisfying (5) for geodesic motion in the Einstein static spacetimes and this becomes a TIR of the spectrum generating algebra for Kepler motion since the corresponding Kepler Hamiltonian is a function of that for the geodesic motion problem. However, the $H = E = 0$ state causes some difficulties. In this paper, we show that it is the existence of a TDR of the $so(4, 2)$ spectrum generating algebra for null geodesic motion in the Einstein static spacetimes (originating from the $so(4, 2)$ algebra of first integrals) which determines the corresponding spectrum generating algebra structure in the classical Kepler problem via equation (8). We address the case of zero energy directly: the integration $\tau(\lambda)$ for $E = 0$ is carried out separately from that for $E \neq 0$ and as such, only provides a set of configurational invariants for the $E = 0$ energy surface. Further, only the subset of time-independent first integrals of motion for $E = 0$ form a Lie algebra. We explain why Cordani's canonical transformation [54] is inadequate for the case of zero energy—this is because the new time coordinate is a first integral for null geodesic motion in Minkowski spacetime ($k = 0$) and a constant of motion cannot be a time coordinate!

We summarize the new results: we present a map from the Einstein static spacetimes which gives time-dependent first integrals of motion for non-zero energies, time-dependent configurational invariants for zero energy, TDR and TIR of $so(4, 2)$ spectrum generating algebra for non-zero energies only, $iso(3)$ invariance algebra for zero energy and an explanation of the inadequacy of the time coordinate (3.15) in [54] for zero energy.

In section 2 we briefly outline some of the concepts necessary for the description of Hamiltonian systems. We outline some results on conformal symmetries of Riemannian and Lorentzian manifolds in section 3 for use in later sections. In this section, we also consider the conformal invariance of null geodesics and the resulting conservation laws in curved spacetimes. We consider the Einstein static spacetimes in section 4 and outline the conformal symmetry properties of these spacetimes. We then specialize to the case of null geodesic motion in Einstein static spacetimes in section 4.2, and in section 4.3 we implement a canonical transformation on the eight-dimensional phase space allowing us to relate this system to the classical Kepler problem. In particular, the transformation relates the time coordinates of the two systems. In section 5.1 we obtain the spectrum generating algebra for the classical Kepler problem for non-zero values of energy $E = H$ from the conformal Killing vector first integrals of null geodesic motion in the Einstein static spacetimes and we verify that we recover the results of [2, 43]. In section 5.2 we deal with the zero energy states: we use an appropriate parameter transformation which allows us to relate the Kepler time to the time coordinate in Minkowski spacetime. In section 5.3 we present the associated TIRs obtained from the TDRs by setting time equal to zero and in the appendix an alternative interpretation of the TIR is given. Finally, in section 6 we give an overview and outline the significance of the results.

2. Hamiltonian systems and Lie algebras

A $2n$-dimensional symplectic manifold N^{2n} endowed with a closed nondegenerate symplectic 2-form $\tilde{\omega}$ is denoted by $(N, \tilde{\omega})$. For a symplectic manifold $(N, \tilde{\omega})$, the Hamiltonian vector field $\hat{\mathbf{X}}^n_f$ corresponding to the function $f : N \mapsto \mathbb{R}$ is defined as the unique smooth vector field on N satisfying

$$\tilde{\omega}(\hat{\mathbf{X}}^n_f) = -\mathbf{d}f.$$

The manifold N^{2n} is described locally by the coordinates $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ and we can write $\tilde{\omega} = \mathbf{d}y_i \wedge \mathbf{d}x^i$. It follows that the Hamiltonian vector field corresponding to the function f has the form

$$\hat{\mathbf{X}}^n_f = \frac{\partial f}{\partial y_\mu}\frac{\partial}{\partial x^\mu} - \frac{\partial f}{\partial x^\nu}\frac{\partial}{\partial y_\nu}. \tag{9}$$

We can define the following operation for two functions $f, g : N \mapsto \mathbb{R}$:

$$\langle f, g\rangle^n = \hat{\mathbf{X}}^n_f(g). \tag{10}$$

A transformation ϕ: $N \mapsto N$ which leaves $\tilde{\omega}$ invariant $\phi^*\tilde{\omega} = \tilde{\omega}$ is said to be canonical. Note that from now on we shall drop the dimension superscripts n to prevent the notation from becoming cluttered, and we shall re-introduce it when appropriate. The Hamiltonian vector fields are the infinitesimal generators of such transformations, that is the Lie derivative of $\tilde{\omega}$ with respect to the Hamiltonian vector field $\hat{\mathbf{X}}_f$ is zero,

$$\mathcal{L}_{\hat{\mathbf{X}}_f}\tilde{\omega} = 0 \tag{11}$$

i.e. the integral curve of the Hamiltonian vector field $\hat{\mathbf{X}}_f$ preserves $\tilde{\omega}$.

Consider the direct product space $W = \mathbb{R} \times N$ which is a $(2n+1)$-dimensional manifold locally described by the coordinates $(x^1, \ldots, x^n, y_1, \ldots, y_n, \lambda)$. Let $H : W \mapsto \mathbb{R}$ with $\mathbf{d}H \neq 0$ be the Hamiltonian function on W. Then we can define a closed 2-form on W by

$$\tilde{\omega}_H = \mathbf{d}y_i \wedge \mathbf{d}x^i - \mathbf{d}H \wedge \mathrm{d}\lambda. \tag{12}$$

Then $(W, \tilde{\omega}_H)$ is said to be an evolution space [5]. Define

$$\hat{\mathbf{Z}}_H = \frac{\partial}{\partial\lambda} + \hat{\mathbf{X}}_H. \tag{13}$$

Then

$$\mathcal{L}_{\hat{\mathbf{Z}}_H}\tilde{\omega}_H = 0 \tag{14}$$

i.e. the integral curve of $\hat{\mathbf{Z}}_H$ preserves $\tilde{\omega}_H$ [5]. The Poisson bracket of two functions $f, g : W \mapsto \mathbb{R}$ is given by

$$\{f, g\}(x, y, ; \lambda) = \langle f_\lambda, g_\lambda\rangle(x, y). \tag{15}$$

The Hamiltonian function H is the function which gives the canonical equations of motion in the given phase space $(N, \tilde{\omega})$. In this paper, we shall consider time-independent Hamiltonian functions $H(x, y)$. A Hamiltonian system is a symplectic manifold $(N, \tilde{\omega})$ endowed with such a Hamiltonian function H and denoted by $(N, \tilde{\omega}, H)$. The Hamiltonian vector field

$$\hat{\mathbf{X}}_H \tag{16}$$

represents the phase flow from which one can read the Hamilton equations of motion.

2.1. First integrals

A non-constant function $C(x, y; \lambda) : W \mapsto \mathbb{R}$ is a first integral of motion if

$$\frac{\mathrm{d}C}{\mathrm{d}\lambda} = \hat{\mathbf{Z}}_H(C) = 0. \tag{17}$$

This is equivalent to equation (1). Note that time-independent first integrals satisfy $\hat{\mathbf{X}}_H(C) = 0$. Poisson's theorem states that the Poisson bracket of two first integrals is a first integral: for any two functions f and g we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\{f, g\} = \left\{\frac{\mathrm{d}f}{\mathrm{d}\lambda}, g\right\} + \left\{f, \frac{\mathrm{d}g}{\mathrm{d}\lambda}\right\} \tag{18}$$

and the result follows from the fact that $\mathrm{d}f/\mathrm{d}\lambda$ and $\mathrm{d}g/\mathrm{d}\lambda$ are identically zero. The first integrals form a Lie algebra under Poisson bracket operation. Note that, in general, such a Lie algebra need not be of finite dimension.

2.2. Configurational invariants

Let us now consider the *configurational invariants* [72–74] $R_I(x, y; \lambda) : W \mapsto \mathbb{R}$ satisfying

$$\hat{\mathbf{Z}}_H(R_I) = HF_I \tag{19}$$

where F_I is an arbitrary function. The R_I are only constants of motion for $H = 0$. When referring to a set of quantities as configurational invariants, it shall be understood that there may be a subset of true first integrals, i.e. satisfying (17) for an arbitrary value of H. Let us now investigate the associated Poisson bracket relations amongst the R_I. Equation (18) then reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\{R_I, R_J\} &= \{HF_I, R_J\} + \{R_I, HF_J\} \\ &= H(\{F_I, R_J\} + \{R_I, F_J\}) + F_I\{H, R_J\} + F_J\{R_I, H\} \\ &= H(\{F_I, R_J\} + \{R_I, F_J\}) + F_J\frac{\partial R_I}{\partial\lambda} - F_I\frac{\partial R_J}{\partial\lambda}\end{aligned} \tag{20}$$

the last line coming from relation (19). The first term vanishes on $H = 0$ hypersurfaces. However, the remaining term is not necessarily equal to zero since the F_I are arbitrary independent functions on the phase space. Thus, the above relation (20) tells us that the Poisson bracket of two configurational invariants is not necessarily a configurational invariant. However, there are special cases: (i) if a set of R_I are time-independent then their Poisson brackets will be time-independent configurational invariants, and so will form a Lie algebra; (ii) if C_I is a time-independent first integral (i.e. $\partial C_I/\partial\lambda = 0$ and $F_I = 0$) then the Poisson bracket with any other configurational invariant R_J is a (possibly time-dependent) configurational invariant. If the Poisson bracket operation does not introduce further new time-dependent configurational invariants then the set will form a Lie algebra. To summarize, if we have a set of time-dependent configurational invariants then *they do not necessarily form a Lie algebra* under the Poisson bracket operation.

2.3. Lie algebras

Consider a function $f(x, y; \lambda) : W \mapsto \mathbb{R}$. If $\hat{\mathbf{Z}}_H(f) = g$ and $H(x, y)$ is a time-independent Hamiltonian function then it follows that $\hat{\mathbf{Z}}_H(\partial f/\partial\lambda) = \partial g/\partial\lambda$. In particular, if $C(x, y; \lambda)$ is a first integral then $\partial C/\partial\lambda$ is a first integral, and if $R(x, y; \lambda)$ is a configurational invariant then $\partial R/\partial\lambda$ is a configurational invariant. If we have a Lie algebra of first integrals C_I then equation (5) gives

$$\frac{\partial}{\partial\lambda}C_I(x, y; \lambda) = D_I^J C_J(x, y; \lambda) \tag{21}$$

and the solution set has the form [2]

$$C_I(x, y; \lambda) = (\mathrm{e}^{D\lambda})_I^J C_J(x, y; \lambda = 0). \tag{22}$$

It follows from definition (15) that the equal time Poisson brackets for $C_I(x, y; \lambda = 0)$ are identical to those for $C_I(x, y; \lambda)$. Thus, given the quantities $C_I(x, y; \lambda)$ one can determine $C_I(x, y; \lambda = 0)$ and conversely, given $C_I(x, y; \lambda = 0)$ one can determine $C_I(x, y; \lambda)$. However, note that for time-dependent quantities C_I, the corresponding quantities $C_I(x, y; \lambda = 0)$ are not necessarily first integrals (in fact, only the time-independent first integrals will remain so). As was stated in the introduction, the first integrals of motion $C_I(x, y; \lambda)$ and $C_I(x, y; \lambda = 0)$ constitute the TDR and TIRs of the spectrum generating algebra, respectively, of the Hamiltonian system.

Now we shall state the relationship between the Lie algebra structures on the configuration space and the phase space. Let N^{2n} be the $2n$-dimensional phase space, or cotangent bundle, corresponding to the n-dimensional configuration space manifold M^n, i.e. $N^{2n} = T^*M^n$. Now consider a Lie algebra of vector fields $\mathbf{Y}_I$ with structure constants D_{IJ}^K on the configuration space M^n, i.e.

$$[\mathbf{Y}_I, \mathbf{Y}_J] = D_{IJ}^K \mathbf{Y}_K. \tag{23}$$

Then it follows that the scalar quantities $\mathcal{Y}_I = Y_I^i y_i$ have the Poisson brackets

$$\{\mathcal{Y}_I, \mathcal{Y}_J\} = D_{IJ}^K \mathcal{Y}_K \tag{24}$$

and that the corresponding Hamiltonian vector fields $\hat{\mathbf{X}}_{\mathcal{Y}_I}$ on T^*M^n share the same structure constants, i.e.

$$\left[\hat{\mathbf{X}}_{\mathcal{Y}_I}, \hat{\mathbf{X}}_{\mathcal{Y}_J}\right] = D_{IJ}^K \hat{\mathbf{X}}_{\mathcal{Y}_K}. \tag{25}$$

2.4. Homogeneous Hamiltonian systems

Given a $2n$-dimensional Hamiltonian system $(T^*\Gamma, \tilde{\omega}, H)$, $H = H(x^i, y_j)$, $i, j = 1, \ldots, n$, then one can construct a corresponding *homogeneous* $2(n+1)$-dimensional Hamiltonian system $(T^*M, \tilde{\Omega}, \mathcal{H})$ where $T^*M = T^*\mathbb{R} \times T^*\Gamma$ and $\mathcal{H} = y_0 + H = 0$. The flow on T^*M then corresponds to the flow on $T^*\Gamma$ given by H [64], i.e. the Hamiltonian vector field corresponding to $\mathcal{H}$ on T^*M is

$$\hat{\mathbf{X}}_{\mathcal{H}} = \hat{\mathbf{X}}_{y_0} + \hat{\mathbf{X}}_H = \frac{\partial}{\partial x^0} + \hat{\mathbf{X}}_H = \hat{\mathbf{Z}}_H. \tag{26}$$

The reverse procedure will be termed *reduction*. In addition, the Poisson brackets $\{C_I, C_J\}^{n+1}$ on T^*M correspond to those $\{C_I, C_J\}^n$ on $T^*\Gamma$. The Poisson brackets for the quantities C_I in $(T^*M, \tilde{\Omega})$ can be written as

$$\{C_I, C_J\}^{n+1} = \{C_I, C_J\}^0 + \{C_I, C_J\}^n \tag{27}$$

where

$$\{C_I, C_J\}^0 = \frac{\partial C_I}{\partial y_0}\frac{\partial C_J}{\partial x^0} - \frac{\partial C_I}{\partial x^0}\frac{\partial C_J}{\partial y_0} \qquad \{C_I, C_J\}^n = \frac{\partial C_I}{\partial y_i}\frac{\partial C_J}{\partial x^i} - \frac{\partial C_I}{\partial x^j}\frac{\partial C_J}{\partial y_j}. \tag{28}$$

Now, on $T^*\Gamma$ consider the case where $C_I = C_I(x^i, y_j, f; \lambda)$ where $f = f(x^k, y_l)$. Then

$$\{C_I, C_J\}^n = \{C_I, C_J\}^n|_f + \{C_I, f\}^n \left.\frac{\partial C_J}{\partial f}\right|_{xy} - \{C_J, f\}^n \left.\frac{\partial C_I}{\partial f}\right|_{xy}. \tag{29}$$

Now, if we choose $f = y_0 = -H$ then from equation (17) we have $\{C_I, f\}^n = -\partial C_I/\partial x^0$, $\lambda = x^0$ and so

$$\{C_I, C_J\}^n = \{C_I, C_J\}^n|_{x^0 y_0} - \left.\frac{\partial C_I}{\partial x^0}\frac{\partial C_J}{\partial y_0}\right|_{xy} + \left.\frac{\partial C_J}{\partial x^0}\frac{\partial C_I}{\partial y_0}\right|_{xy} \equiv \{C_I, C_J\}^{n+1} \tag{30}$$

where $C_I = C_I(x^i, y_j, x^0, y_0)$. Thus the Poisson bracket commutation relations between the first integrals of motion on the $2(n+1)$-dimensional Hamiltonian system $(T^*M, \tilde{\Omega}, \mathcal{H})$ are identical to those for the equal time Poisson brackets for the corresponding $2n$-dimensional system $(T^*\Gamma, \tilde{\omega}, H)$.

3. Conformal symmetries of configuration space

Consider an n-dimensional manifold M with (Riemannian or Lorentzian) metric tensor $\mathbf{g}$. Let $\mathbf{y}$ be an arbitrary geodesic tangent vector and $\xi^i = \xi^i(x^j)$ an arbitrary vector field. Then $\xi^i y_i$ is the component of the vector field ξ along the geodesic tangent vector $\mathbf{y}$. We can investigate the variation of the quantities $\xi^i y_i$ along such a geodesic. It is straightforward to show that for an arbitrary vector field ξ

$$\hat{\mathbf{X}}_G(\xi^i y_i) = (\mathcal{L}_\xi \mathbf{g})(\mathbf{y}, \mathbf{y})/2 \tag{31}$$

where G is the Hamiltonian function $G = \mathbf{g}(\mathbf{y}, \mathbf{y})/2$ and $\mathcal{L}_\xi$ denotes the Lie derivative with respect to ξ.

A transformation $\Phi: M \mapsto M$ such that $\Phi: \mathbf{g} \mapsto \psi(x^i)\mathbf{g}$ is called a *conformal transformation* and the set of such transformations forms a group. The subset of continuous transformations forms a Lie group and the corresponding infinitesimal generators form a Lie algebra. The infinitesimal generators ξ of conformal transformations on M are referred to as *conformal Killing vector fields* (CKVs) and satisfy

$$\mathcal{L}_\xi \mathbf{g} = 2\phi(x^i)\mathbf{g}. \tag{32}$$

If the function $\phi = constant$ for some vector field ξ then ξ is called a *homothetic Killing vector field* (HKV) and if $\phi = 0$ then ξ is called a *Killing vector field* (KV). The HKVs form a subalgebra as do the KVs. It follows from (31) that if ξ is a CKV then

$$\hat{\mathbf{X}}_G(\xi^i y_i) = 2\phi G \tag{33}$$

Thus, if ξ is a KV then $\xi^i y_i$ is a first integral of geodesic motion. If the CKV ξ is not a KV, then $\xi^i y_i$ is conserved in the case of $G = 0$ only, i.e. $\xi^i y_i$ is a configurational invariant satisfying (19). However, these are time parameter independent and so their Poisson bracket is a configurational invariant and the set has a Lie algebra structure.

Suppose a manifold with metric $\mathbf{g}$ admits a CKV ξ. Then any conformally related space with metric tensor $\tilde{\mathbf{g}} = \Omega^2(x^i)\mathbf{g}$ also admits ξ as a CKV. We can investigate the Lie derivative of this metric tensor with respect to the conformal Killing vectors of $\mathbf{g}$. We have that

$$\mathcal{L}_\xi(\tilde{\mathbf{g}}) = \mathcal{L}_\xi(\Omega^2)\mathbf{g} + \Omega^2\mathcal{L}_\xi(\mathbf{g}) = 2\tilde{\phi}\Omega^2\mathbf{g} \tag{34}$$

where $\tilde{\phi} = [\xi(\ln\Omega) + \phi]$, i.e. a CKV ξ of $\mathbf{g}$ is necessarily a CKV of $\tilde{\mathbf{g}}$ with conformal factor $\tilde{\phi}$.

Now consider a four-dimensional spacetime manifold M with metric tensor $\mathbf{g}$ of Lorentz signature. If the CKV ξ is not a KV, then $\xi^i y_i$ is conserved in the case of null geodesics only. Thus, writing null geodesic motion in terms of the Hamiltonian function $\mathcal{G} = \mathbf{g}(\mathbf{y}, \mathbf{y})/2 = 0$, we have from equation (17) that $R = \xi^i y_i$ are configurational invariants for the system, i.e.

$$\hat{\mathbf{X}}^4_{\mathcal{G}}(R) = 2\phi\mathcal{G}. \tag{35}$$

Of course, since R are independent of the time parameter, by remark (i) in section 2.2, they form a Lie algebra. The null geodesics are conformally invariant: consider a spacetime with metric tensor $\mathbf{g}$, then geodesic motion can be written in terms of the Hamiltonian function $\mathcal{G} = \mathbf{g}(\mathbf{y}, \mathbf{y})/2$. Now consider a conformally related spacetime with metric tensor $\tilde{\mathbf{g}} = \Omega^2(x^i)\mathbf{g}$. It follows that the corresponding Hamiltonian function is given by $\tilde{\mathcal{G}} = \Omega^{-2}\mathcal{G}$ and geodesic motion on the new spacetime is given by the flow

$$\hat{\mathbf{X}}^4_{\tilde{\mathcal{G}}} = \Omega^{-2}\hat{\mathbf{X}}^4_{\mathcal{G}} + 2\mathcal{G}\Omega^{-3}\frac{\partial\Omega}{\partial x^i}\frac{\partial}{\partial y_i}. \tag{36}$$

Thus, in general the two Hamiltonian vector fields corresponding to $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are not parallel. However, when we consider null geodesics ($\mathcal{G} = 0$) the last term vanishes and the Hamiltonian vector fields become parallel. Upon a change of parameter $\mathrm{d}\tilde{\lambda} = \Omega^2\,\mathrm{d}\lambda$ the null geodesics are mapped into null geodesics. It follows that the null geodesic structure is preserved under conformal transformations.

4. Einstein static spacetimes

4.1. Geometry

In the following, Greek indices take the values 0, 1, 2, 3, Latin indices take the values 1, 2, 3 and the Einstein summation rule is assumed unless otherwise indicated. In this and the following sections $a\cdot b \equiv a^i b_i$ and $|a|^2 \equiv \delta_{ij}a^i a^j$. Let (Γ, g) be a three-dimensional manifold of Euclidean signature with constant curvature k. We shall refer to the four-dimensional product space $M = \mathbb{R}\times\Gamma$ of Lorentz signature as an *Einstein static spacetime* $(M, \mathfrak{G})$. The manifold Γ is described locally by the coordinates x^i and the manifold M by the coordinates $x^\alpha \equiv (x^0, x^i)$. The three-dimensional space Γ is conformally Euclidean, the line element being

$$\mathrm{d}s^2 = K_+^{-2}\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{37}$$

where $K_\pm = (1 \pm k|\mathbf{x}|^2/4)$ and k is a constant. The four-dimensional Einstein static spacetime $(M, \mathfrak{G})$ is conformally Minkowskian. The line element for $(M, \mathfrak{G})$ has the form

$$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + K_+^{-2}\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{38}$$

However, we have chosen not to write the line element in a manifestly conformally Minkowskian form since the form above is more convenient for our present purposes—see [70] for the transformation $x^\alpha = x^\alpha(z^\beta)$ required to put (38) in a conformally Minkowskian form.

Minkowski spacetime has the line element

$$\mathrm{d}s^2 = \eta_{\alpha\beta}\,\mathrm{d}z^\alpha\,\mathrm{d}z^\beta = -(\mathrm{d}z^0)^2 + \delta_{ij}\,\mathrm{d}z^i\,\mathrm{d}z^j. \tag{39}$$

The conformal symmetry group of Minkowski spacetime is the 15-parameter *conformal group* $SO(4, 2)$ and includes the isometry group $ISO(3, 1)$. The 15 CKVs are

$$\begin{aligned} &\mathbf{T}_\alpha = \frac{\partial}{\partial z^\alpha} && \mathbf{M}_{\alpha\beta} = z_\alpha\frac{\partial}{\partial z^\beta} - z_\beta\frac{\partial}{\partial z^\alpha} \\ &\mathbf{D} = z^\alpha\frac{\partial}{\partial z^\alpha} && \mathbf{K}_\alpha = 2z_\alpha z^\beta\frac{\partial}{\partial z^\beta} - (z^\beta z_\beta)\frac{\partial}{\partial z^\alpha} \end{aligned} \tag{40}$$

where $z_\alpha = \eta_{\alpha\beta}z^\beta$ and we shall refer to this as the Minkowski basis. The commutation relations for the vector fields (40) are as follows

$$\begin{aligned} &[\mathbf{M}_{\alpha\beta}, \mathbf{M}_{\gamma\delta}] = \eta_{\alpha\delta}\mathbf{M}_{\beta\gamma} + \eta_{\beta\gamma}\mathbf{M}_{\alpha\delta} + \eta_{\alpha\gamma}\mathbf{M}_{\delta\beta} + \eta_{\beta\delta}\mathbf{M}_{\gamma\alpha} \\ &[\mathbf{T}_\alpha, \mathbf{T}_\beta] = 0 \qquad [\mathbf{T}_\alpha, \mathbf{M}_{\beta\gamma}] = \eta_{\beta\alpha}\mathbf{T}_\gamma - \eta_{\gamma\alpha}\mathbf{T}_\beta \\ &[\mathbf{K}_\alpha, \mathbf{K}_\beta] = 0 \qquad [\mathbf{K}_\alpha, \mathbf{M}_{\beta\gamma}] = \eta_{\beta\alpha}\mathbf{K}_\gamma - \eta_{\gamma\alpha}\mathbf{K}_\beta \\ &[\mathbf{D}, \mathbf{K}_\alpha] = \mathbf{K}_\alpha \qquad [\mathbf{D}, \mathbf{M}_{\alpha\beta}] = 0 \qquad [\mathbf{T}_\alpha, \mathbf{D}] = \mathbf{T}_\alpha \\ &[\mathbf{T}_\alpha, \mathbf{K}_\beta] = 2(\eta_{\alpha\beta}\mathbf{D} - \mathbf{M}_{\alpha\beta}) \end{aligned} \tag{41}$$

Table 1. The conformal scalars for the CKV of Einstein static spacetime $(M, \mathfrak{G})$.

Vector field	Conformal scalar	Type
$\mathbf{T}_0$	$\phi = -k\epsilon K_- K_+^{-1}/2$	CKV
$\mathbf{T}_i$	$\phi = -kK_+^{-1}\theta x^i/2$	CKV
$\mathbf{M}_{0i}$	$\phi = kK_+^{-1}\epsilon x^i$	CKV
$\mathbf{M}_{ij}$	$\phi = 0$	KV
$\mathbf{D}$	$\phi = K_- K_+^{-1}\theta$	CKV
$\mathbf{K}_0$	$\phi = -2\epsilon K_- K_+^{-1}$	CKV
$\mathbf{K}_i$	$\phi = 2K_+^{-1}\theta x^i$	CKV

which is isomorphic to the Lie algebra $so(4, 2)$ [70]. The ten KVs $\mathbf{T}_\alpha$ and $\mathbf{M}_{\beta\gamma}$ form an $iso(3,1)$ isometry subalgebra.

Since the Einstein static spacetimes are conformally Minkowskian, they are conformally invariant under the action of the conformal group $SO(4, 2)$. The 15 CKVs for the Einstein static spacetimes (38) for arbitrary values of curvature k can be derived by constructing a direct map from Minkowski spacetime [70]. In the following, the quantity ϵ is defined by the relation $\mathrm{d}x^0 = \mathrm{d}\epsilon/\theta(\epsilon)$ where $\theta(\epsilon) = (1 - k\epsilon^2)^{\frac{1}{2}}$. The 15 CKVs can be written as [70]

$$\begin{aligned}
&\mathbf{T}_0 = \tfrac{1}{2}(\mathbf{V}_0 + \mathbf{E}) \qquad \mathbf{T}_i = \tfrac{1}{2}(\mathbf{V}_i + \mathbf{S}_i)\\
&\mathbf{M}_{0i} = -\theta K_+^{-1} x^i \frac{\partial}{\partial x^0} - \epsilon\left(-\mathbf{V}_i + 2\frac{\partial}{\partial x^i}\right) \qquad \mathbf{M}_{ij} = x^i\frac{\partial}{\partial x^j} - x^j\frac{\partial}{\partial x^i}\\
&\mathbf{D} = \epsilon K_- K_+^{-1}\frac{\partial}{\partial x^0} + \theta\left(x^j\frac{\partial}{\partial x^j}\right)\\
&\mathbf{K}_0 = -\frac{4\epsilon}{\left(\theta + K_- K_+^{-1}\right)}\mathbf{D} - \frac{4}{\left(\theta + K_- K_+^{-1}\right)^2}\left(-\epsilon^2 + |x|^2 K_+^{-2}\right)\mathbf{T}_0\\
&\mathbf{K}_i = \frac{4K_+^{-1}x^i}{\left(\theta + K_- K_+^{-1}\right)}\mathbf{D} - \frac{4}{\left(\theta + K_- K_+^{-1}\right)^2}\left(-\epsilon^2 + |x|^2 K_+^{-2}\right)\mathbf{T}_i
\end{aligned} \tag{42}$$

where

$$\begin{aligned}
&\mathbf{V}_0 = \frac{\partial}{\partial x^0} \qquad \mathbf{V}_i = K_-\frac{\partial}{\partial x^i} + \frac{k}{2}x^i\left(x^j\frac{\partial}{\partial x^j}\right)\\
&\mathbf{E} = K_- K_+^{-1}\theta\frac{\partial}{\partial x^0} - k\epsilon\left(x^j\frac{\partial}{\partial x^j}\right) \qquad \mathbf{S}_i = -k\epsilon K_+^{-1}x^i\frac{\partial}{\partial x^0} + \theta\left(-\mathbf{V}_i + 2\frac{\partial}{\partial x^i}\right).
\end{aligned}$$

The corresponding conformal scalars are given in table 1. It is important to note that the vector fields (42) have Lie brackets (41), i.e. the structure constants are *independent* of the value of k. Note that for $k = 0$, $x^\alpha = z^\alpha$ and we get the Minkowski CKV (41). We can choose an alternative basis consisting of $\mathbf{M}_{0i}, \mathbf{M}_{ij}, \mathbf{D}, \mathbf{V}_0, \mathbf{V}_i, \mathbf{E}, \mathbf{S}_i$ for the case where $k \neq 0$, for $k = 0$ these do not span a 15-dimensional vector space and so has to be excluded (see section 5.1 of [70]). When $k \neq 0$ there is a seven-dimensional KV subalgebra with basis $\mathbf{M}_{ij}$, $\mathbf{V}_i$ and $\mathbf{V}_0$, and a further six-dimensional KV subalgebra with basis $\mathbf{M}_{ij}$, $\mathbf{V}_i$ and the latter is isomorphic to $so(4)$ and $so(3, 1)$ for $k > 0$ and $k < 0$ respectively. We now present the 15 associated quantities $R_I = \xi_I^\beta y_\beta$ which constitute a basis for the Poisson bracket Lie algebra of configurational invariants, i.e. first integrals for null geodesics (see section 4.2). We shall write $\mathcal{T}_\alpha = (\mathbf{T}_\alpha)^\beta y_\beta$ etc. They are as follows,

$$\begin{aligned}
&\mathcal{T}_0 = \tfrac{1}{2}(\mathcal{V}_0 + \mathcal{E}) \qquad \mathcal{T}_i = \tfrac{1}{2}(\mathcal{V}_i + \mathcal{S}_i) \\
&\mathcal{M}_{0i} = -\theta K_+^{-1} x^i y_0 - \epsilon(-\mathcal{V}_i + 2y_i) \qquad \mathcal{M}_{ij} = x^i y_j - x^j y_i \\
&\mathcal{D} = \epsilon K_- K_+^{-1} y_0 + \theta(x \cdot y) \\
&\mathcal{K}_0 = -\frac{4\epsilon}{\left(\theta + K_- K_+^{-1}\right)}\mathcal{D} - \frac{4}{\left(\theta + K_- K_+^{-1}\right)^2}\left(-\epsilon^2 + |x|^2 K_+^{-2}\right)\mathcal{T}_0 \\
&\mathcal{K}_i = \frac{4K_+^{-1}x^i}{\left(\theta + K_- K_+^{-1}\right)}\mathcal{D} - \frac{4}{\left(\theta + K_- K_+^{-1}\right)^2}\left(-\epsilon^2 + |x|^2 K_+^{-2}\right)\mathcal{T}_i
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
&\mathcal{V}_0 = y_0 \qquad \mathcal{V}_i = K_- y_i + \frac{k}{2} x^i (x \cdot y) \\
&\mathcal{E} = K_- K_+^{-1} \theta y_0 - k\epsilon(x \cdot y) \qquad \mathcal{S}_i = -k\epsilon K_+^{-1} x^i y_0 + \theta(-\mathcal{V}_i + 2y_i).
\end{aligned}$$

Equation (24) implies that the Poisson bracket Lie algebra of these configurational invariants is identical to that for the corresponding CKV, i.e. the Lie algebras have the same structure constants. We note that, amongst these first integrals of null geodesic motion (which are independent of the parameter λ), including the Hamiltonian, there can be at most seven which are functionally independent [71, 75]. Thus, once the value of the Hamiltonian has been specified (i.e. $\mathcal{G} = 0$) there can be only six other functionally independent first integrals. We note that when $k \neq 0$ we can write

$$\epsilon = \frac{1}{\sqrt{k}} \sin(\sqrt{k} x^0) \qquad \theta(\epsilon) = \cos(\sqrt{k} x^0) \qquad k > 0 \tag{44}$$

$$\epsilon = \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k} x^0) \qquad \theta(\epsilon) = \cosh(\sqrt{-k} x^0) \qquad k < 0. \tag{45}$$

4.2. Null geodesic motion

Let $(T^*\Gamma, \tilde{\omega})$ be the six-dimensional cotangent bundle associated with Γ. Further, let (x^i, y_j) be local coordinates on $(T^*\Gamma, \tilde{\omega})$ and $(x^\alpha, y_\beta) \equiv (x^0, x^i, y_0, y_j)$ be local coordinates on the eight-dimensional cotangent bundle $(T^*M, \tilde{\Omega})$ where $\tilde{\Omega}$ is the symplectic 2-form $\tilde{\Omega} = \tilde{\omega} + \mathrm{d}y_0 \wedge \mathrm{d}x^0$.

Consider an Einstein static spacetime $(M, \mathfrak{G})$. Then geodesic motion in such a spacetime can be represented in the eight-dimensional phase space by the Hamiltonian system $(T^*M, \tilde{\Omega}, \mathcal{G})$ where the Hamiltonian function $\mathcal{G}$ is defined as

$$\mathcal{G} = -\tfrac{1}{2} y_0^2 + G \qquad G = K_+^2 |\mathbf{y}|^2/2. \tag{46}$$

The geodesic phase flow is represented on the evolution space $\mathbb{R} \times T^*M$ by the vector field

$$\hat{\mathbf{Z}}^4_{\mathcal{G}} \tag{47}$$

and since the 15 configurational invariants $R_I(x^\alpha, y_\beta)$ given in (43) have no explicit parameter dependence,

$$\frac{\mathrm{d}R_I}{\mathrm{d}\lambda} = \hat{\mathbf{X}}^4_{\mathcal{G}}(R_I) = 2\phi\mathcal{G}. \tag{48}$$

Of course, the KVs ($\phi = 0$) yield first integrals for arbitrary values of the Hamiltonian $\mathcal{G}$. The $R_I(x^\alpha, y_\beta)$ are all first integrals for null geodesic motion $\mathcal{G} = 0$.

We can consider the geodesic motion on (Γ, g) as a reduction (see section 2.4) of the null geodesic motion on $(M, \mathfrak{G})$. This immediately provides us with a basis for the 15-dimensional spectrum generating algebra for the system $(T^*\Gamma, \tilde{\omega})$. First, we implement the trivial canonical transformation

$$x'^0 = -x^0/y_0 \qquad x'^i = x^i \qquad y'_0 = -(y_0)^2/2 \qquad y'_j = y_j.$$

It follows that

$$\mathcal{G} = y'_0 + G \tag{49}$$

and

$$\hat{\mathbf{X}}^4_{\mathcal{G}} = -y_0\frac{\partial}{\partial x^0} + \hat{\mathbf{X}}^3_G = \frac{\partial}{\partial x'^0} + \hat{\mathbf{X}}^3_G = \hat{\mathbf{Z}}^3_G = \frac{\mathrm{d}}{\mathrm{d}x'^0}.$$

Second, by restricting to null geodesic motion on the Einstein static spacetime i.e. $(T^*M, \tilde{\Omega}, \mathcal{G} = 0)y_0$ is replaced everywhere by $y_0 = y_0(x^i, y_j, x^0)$ and we have that the quantities $C_I(x^i, y_j, x^0)$ are 15 time (x'^0) dependent first integrals for geodesic motion on a three-dimensional space of constant curvature k, i.e. the R_I for $(T^*M, \tilde{\Omega}, \mathcal{G})$ become C_I for $(T^*\Gamma, \tilde{\omega}, G)$. It then follows from (30) that the spectrum generating algebra is isomorphic to $so(4, 2)$. Note that $C_I(x^i, y_j, x^0 = 0)$ form the associated TIR. Thus we obtain a basis for the spectrum generating algebra for geodesic motion on the three-dimensional space Γ of constant curvature k. Dothan (section V.A. of [2]) presents the ten-dimensional subalgebra for the case of geodesic motion in flat Euclidean space $k = 0$.

Let us now consider the TIRs of the spectrum generating algebras associated with geodesic motion on Γ. Quantities (43) with $x^0 = 0$ form the TIR for geodesic motion on the space Γ of constant curvature k for all values of k: we present an alternative basis composed of the 15 quantities $\mathcal{M}_{0i}, \mathcal{M}_{ij}, \mathcal{D}, \mathcal{V}_0, \mathcal{V}_i, \mathcal{E}, \mathcal{S}_i$ which is valid for non-zero values of k only,

$$\begin{aligned} &\mathcal{M}_{0i} = -|y|x^i \qquad \mathcal{M}_{ij} = x^i y_j - x^j y_i \qquad \mathcal{D} = (x \cdot y) \qquad \mathcal{V}_0 = K_+|y| \\ &\mathcal{V}_i = K_- y_i + k(x \cdot y)x^i/2 \qquad \mathcal{E} = K_-|y| \qquad \mathcal{S}_i = -\mathcal{V}_i + 2y_i. \end{aligned} \tag{50}$$

From the remarks made in section 2.3, these 15 quantities still constitute a basis for the Lie algebra $so(4, 2)$ and we see that $\mathcal{V}_0$ plays the role of a Hamiltonian $\sqrt{2G}$. For a fixed value of k, this spectrum generating algebra allows transitions between states with different values of $G > 0$.

In the following sections, we use a similar method to obtain the spectrum generating algebra for the classical Kepler problem. The difference lies in the properties of the canonical transformation used.

4.3. Canonical transformations on $(T^*M, \tilde{\Omega})$

It has been shown [69] that the Kepler equations of motion can be derived from the equations of geodesic motion on three-dimensional spaces of constant curvature k via the following three transformations: spatial coordinate inversion,

$$\begin{aligned} &x'^i = \frac{x^i}{|x|^2} \qquad x'^0 = x^0 \qquad y'_i = |x|^2 y_i - 2x^i(x \cdot y) \\ &y'_0 = y_0 \qquad \text{(canonical transformation I)} \end{aligned}$$

spatial position/momenta interchange,

$$\begin{aligned} &\bar{x}^i = y'_i/2\sqrt{2} \qquad \bar{x}^0 = x'^0 \qquad \bar{y}_i = -2\sqrt{2}x'^i \\ &\bar{y}_0 = y'_0 \qquad \text{(canonical transformation II)} \end{aligned}$$

and the parameter transformation

$$\mathrm{d}\tau = \mathcal{F}\,\mathrm{d}\lambda \tag{51}$$

where $\mathcal{F} = \sqrt{G}|\bar{x}|$. Note that the general solution to the differential equation (51) can be written in terms of Stumpff functions [64]. However, we use a more direct approach which is equivalent to, but more enlightening than, the integration in [64].

In this section, we show that the parameter transformation (51) can be integrated and extended to give a canonical transformation in the eight-dimensional phase space T^*M. We begin by considering the quantity $\mathfrak{D} = x^0 y_0 + (x \cdot y)$. We find that

$$\hat{\mathbf{X}}^4_{\mathcal{G}}(\mathfrak{D}) = 2\mathcal{G} - 2k\mathcal{F} \tag{52}$$

and so on null surfaces $\mathcal{G} = 0$ we have

$$\frac{\mathrm{d}\mathfrak{D}}{\mathrm{d}\lambda} = -2k\mathcal{F}. \tag{53}$$

At this point, we make a very important observation: for the special case where $k = 0$ the quantity $\mathfrak{D} = \mathcal{D}$, which is a first integral of motion. We can introduce the parameter τ, by equation (51), and it follows that

$$\frac{\mathrm{d}\mathfrak{D}}{\mathrm{d}\tau} = -2k. \tag{54}$$

At this point, we note that equation (54) can be integrated to give

$$-2k\tau = \mathfrak{D} \tag{55}$$

and we take an arbitrary constant to be zero. Thus we can regard the quantity $-\mathfrak{D}/2k$ as a new coordinate for non-zero values of k. $\mathfrak{D}$ cannot be regarded as a time coordinate for $k = 0$ and this case is dealt with separately in section 5.2.

The transformation (55) can be extended to T^*M to give a canonical transformation as follows. (Note that since τ has the status of a canonical coordinate on the eight-dimensional phase space, we re-label it as q^0, and k has the status of a canonical momentum p_0.)

$$\begin{aligned} q^k &= \sqrt{2}\bar{y}_0\bar{x}^k/\alpha & q^0 &= 2[\bar{x}^0\bar{y}_0 + (\bar{x}\cdot\bar{y})](\bar{y}_0/\alpha)^2 \\ p_k &= \alpha\bar{y}_k/\sqrt{2}\bar{y}_0 & p_0 &= -\alpha^2/4\bar{y}_0^2 \qquad \text{(canonical transformation III)} \end{aligned}$$

with inverse

$$\begin{aligned} \bar{x}^k &= -\sqrt{-2p_0}q^k & \bar{x}^0 &= \sqrt{2}\big[-q^0(-2p_0)^{\frac{3}{2}} + (-2p_0)^{\frac{1}{2}}(q\cdot p)\big]/\alpha \\ \bar{y}_0 &= -\alpha/\sqrt{-4p_0} & \bar{y}_k &= -p_k/\sqrt{-2p_0}. \end{aligned}$$

The quantity α is an arbitrary non-zero constant and we note that $p_0 < 0$ necessarily. We emphasize that the canonical transformations I, II and III are *independent* of the value of k. Under canonical transformations I and II the Hamiltonian $\mathcal{G}$ becomes

$$\mathcal{G} = -\frac{1}{2}\bar{y}_0^2 + \frac{1}{4}\left(k + \frac{|\bar{y}|^2}{2}\right)^2 |\bar{x}|^2 \tag{56}$$

and under the canonical transformation III

$$(-8p_0)\mathcal{G} = -\alpha^2 + \left(-2p_0k + \frac{|p|^2}{2}\right)^2 |q|^2. \tag{57}$$

At this point, we can see that $\mathcal{G} = 0$ implies (making a sign choice)

$$\alpha = \left(-2p_0k + \frac{|p|^2}{2}\right)|q|. \tag{58}$$

The Hamiltonian vector field on T^*M corresponding to the function $(-8p_0)\mathcal{G}$ on the LHS of equation (57) is

$$-8\mathcal{G}\hat{\mathbf{X}}^4_{p_0} + (-8p_0)\hat{\mathbf{X}}^4_{\mathcal{G}} \tag{59}$$

and the first term vanishes when $\mathcal{G} = 0$. Ultimately we find, using equations (57) and (58),

$$\hat{\mathbf{X}}^4_{\mathcal{G}} = \frac{\alpha|q|}{(-4p_0)}\left[-2k\frac{\partial}{\partial q^0} + \hat{\mathbf{X}}\right] \tag{60}$$

where

$$\hat{\mathbf{X}} = p_i\frac{\partial}{\partial q^i} - \frac{\alpha}{|q|^3}q^j\frac{\partial}{\partial p_j}. \tag{61}$$

Thus, the Hamiltonian vector field $\hat{\mathbf{X}}^4_{\mathcal{G}}$ corresponding to the flow given by $\mathcal{G}$ is *parallel* to the vector field in the parentheses on the RHS of equation (60). Now it is important to realize that the canonical transformations I, II and III used to achieve this result are *independent* of the value of k and so are valid for all values of k. However, equation (60) reinforces the fact that q^0 is not a time coordinate for the case $k = 0$. It is equation (60) that provides the link with the Kepler problem: $\hat{\mathbf{X}}$ is seen to be the Hamiltonian vector field on $T^*\Gamma$ corresponding to the Kepler motion [69]. It only remains to rescale the time coordinate (for $k \neq 0$) so that the vector field in the parentheses in equation (60) gives the time evolution of the Kepler problem.

5. The classical Kepler problem

5.1. Non-zero energy states

For the case where $k \neq 0$ we can introduce further new canonical coordinates

$$Q^i = q^i \qquad Q^0 = -q^0/2k \qquad P_j = p_j$$
$$P_0 = -2kp_0 \qquad \text{(canonical transformation IV)}$$

and equation (60) becomes

$$\hat{\mathbf{X}}^4_{\mathcal{G}} = \frac{\alpha k|Q|}{2P_0}\left[\frac{\partial}{\partial Q^0} + \hat{\mathbf{X}}^3\right]. \tag{62}$$

Equation (58) gives

$$-P_0 = \frac{|P|^2}{2} - \frac{\alpha}{|Q|} \tag{63}$$

which we immediately recognize as the Hamiltonian for the Kepler problem and accordingly we write $H = -P_0$. Since $p_0 < 0$ necessarily, k has opposite sign to Kepler Hamiltonian $H = +2kp_0$. The Hamiltonian vector field in the parentheses on the RHS of equation (62) is

$$\hat{\mathbf{X}}^4_{\mathcal{H}} = \frac{\partial}{\partial Q^0} + \hat{\mathbf{X}}_H \tag{64}$$

where

$$\mathcal{H} = H + P_0 = 0. \tag{65}$$

Under reduction, the Hamiltonian vector field in (64) is equivalent to

$$\hat{\mathbf{Z}}_H = \frac{\partial}{\partial Q^0} + \hat{\mathbf{X}}_H \tag{66}$$

on the evolution space $\mathbb{R} \times T^*\Gamma$. The fact that the phase flows are parallel means that the flows are equivalent under a change of parameter (noting that $\mathcal{F} = \alpha|Q|/2$),

$$\mathrm{d}\Upsilon = -\frac{k\mathcal{F}}{H}\,\mathrm{d}\lambda \tag{67}$$

and that the first integrals $C_I(x^i, y_j, x^0, y_0)$ for the system $(T^*M, \tilde{\Omega}, \mathcal{G} = 0)$ are first integrals of motion for the system corresponding to phase flow (66). Thus the first integrals $C_I(x^i, y_j, x^0, y_0) \equiv C_I(q^i, p_j, q^0, p_0) \equiv C_I(Q^i, P_j, Q^0, P_0)$ become the first integrals for the system $(T^*M, \tilde{\Omega}, \mathcal{H} = 0)$ and ultimately the first integrals of motion for the Kepler Hamiltonian system $(T^*\Gamma, \tilde{\omega}, H)$ with associated evolution space $(W, \tilde{\omega}_H)$ and time parameter Q^0. Thus the equations of motion for the classical Kepler problem for non-zero values of energy can be derived from those equations of motion corresponding to null geodesic motion in an Einstein static spacetime of non-zero curvature k and the quantities C_I are first integrals of motion for both systems. Transformations I–IV are collectively equivalent to that in [54] except for the presence of the mass parameter α. The $so(4, 2)$ Poisson bracket structure is preserved by the correspondence in equations (23) and (24) in section 2.3, the canonical transformations I–IV and the canonical nature of the reduction procedure in section 2.4.

We now give a basis for the TDR of the spectrum generating algebra of the classical Kepler problem. We simply take expressions (43) and apply the canonical transformations I–IV. We then have

$$
\begin{aligned}
&\mathcal{T}_0 = \tfrac{1}{2}(\mathcal{V}_0 + \mathcal{E}) \qquad \mathcal{T}_i = \tfrac{1}{2}(\mathcal{V}_i + \mathcal{S}_i)\\
&\mathcal{M}_{0i} = \frac{2\alpha\theta}{(|P|^2 - 2H)} P_i - \epsilon[-\mathcal{V}_i + \sqrt{-k/2H}(-|P|^2 Q^i + 2(Q\cdot P)P_i)]\\
&\mathcal{M}_{ij} = Q^i P_j - Q^j P_i\\
&\mathcal{D} = -\alpha\sqrt{-k/2H}\left[\frac{|P|^2 + 2H}{|P|^2 - 2H}\right]\epsilon + \theta(Q\cdot P)\\
&\mathcal{K}_0 = \frac{-4\epsilon(|P|^2 - 2H)}{[\theta(|P|^2 - 2H) + (|P|^2 + 2H)]}\mathcal{D} - 4\frac{[-\epsilon^2(|P|^2 - 2H)^2 - 8H|P|^2/k]}{[\theta(|P|^2 - 2H) + (|P|^2 + 2H)]^2}\mathcal{T}_0\\
&\mathcal{K}_i = \frac{8\sqrt{2}\sqrt{-H/k}P_i}{[\theta(|P|^2 - 2H) + (|P|^2 + 2H)]}\mathcal{D} - 4\frac{[-\epsilon^2(|P|^2 - 2H)^2 - 8H|P|^2/k]}{[\theta(|P|^2 - 2H) + (|P|^2 + 2H)]^2}\mathcal{T}_i
\end{aligned}
\tag{68}
$$

where

$$
\begin{aligned}
&\mathcal{V}_0 = -\alpha\sqrt{-k/2H} \qquad \mathcal{V}_i = \sqrt{-k/2H}[-(|P|^2 + 2H)Q^i + 2(Q\cdot P)P_i]/2\\
&\mathcal{E} = -\alpha\sqrt{-k/2H}\left[\frac{|P|^2 + 2H}{|P|^2 - 2H}\right]\theta - k\epsilon(Q\cdot P)\\
&\mathcal{S}_i = \frac{2\alpha k\epsilon}{(|P|^2 - 2H)} P_i + \theta\big[-\mathcal{V}_i + \sqrt{-k/2H}(-|P|^2 Q^i + 2(Q\cdot P)P_i)\big]
\end{aligned}
\tag{69}
$$

and θ and ϵ are functions of the new time coordinate $\bar{x}^0 = \bar{x}^0(Q^0, Q^i, P_j)$.

Now, we can write these expressions explicitly for the cases of negative and positive energies separately, bearing in mind relations (44), (45) and the functional form for the Kepler Hamiltonian H. For negative energies, we have

$$
\begin{aligned}
&\mathcal{T}_0 = \tfrac{1}{2}(\mathcal{V}_0 + \mathcal{E}) \qquad \mathcal{T}_i = \tfrac{1}{2}(\mathcal{V}_i + \mathcal{S}_i)\\
&\mathcal{M}_{0i} = |Q|P_i\cos\beta - \sqrt{-1/2H}\left[-\frac{\alpha}{|Q|}Q^i + (Q\cdot P)P_i\right]\sin\beta\\
&\mathcal{M}_{ij} = Q^i P_j - Q^j P_i\\
&\mathcal{D} = -\sqrt{-1/2H}(2H|Q| + \alpha)\sin\beta + (Q\cdot P)\cos\beta\\
&\mathcal{K}_0 = \frac{-4\sin\beta}{\sqrt{k}[\cos\beta + (|P|^2 + 2H)|Q|/2\alpha]}\mathcal{D} + \frac{16[\alpha^2\sin^2\beta + 2H|P|^2|Q|^2]}{k[2\alpha\cos\beta + (|P|^2 + 2H)|Q|]^2}\mathcal{T}_0\\
&\mathcal{K}_i = \frac{8\sqrt{-2H}P_i}{[(2\alpha/|Q|)\cos\beta + (|P|^2 + 2H)]}\mathcal{D} + \frac{16[\alpha^2\sin^2\beta + 2H|P|^2|Q|^2]}{k[2\alpha\cos\beta + (|P|^2 + 2H)|Q|]^2}\mathcal{T}_i
\end{aligned}
\tag{70}
$$

where

$$\begin{aligned}
&\mathcal{V}_0 = -\alpha\sqrt{-k/2H} \qquad \mathcal{V}_i = \sqrt{-k/2H}[-(|P|^2 + 2H)Q^i + 2(Q\cdot P)P_i]/2 \\
&\mathcal{E} = -\sqrt{-k/2H}(2H|Q| + \alpha)\cos\beta - \sqrt{k}(Q\cdot P)\sin\beta \\
&\mathcal{S}_i = \sqrt{k}|Q|P_i\sin\beta + \sqrt{-k/2H}\left[-\frac{\alpha}{|Q|}Q^i + (Q\cdot P)P_i\right]\cos\beta
\end{aligned} \tag{71}$$

and β is defined (in accordance with [43]) as

$$\beta = \sqrt{k}\bar{x}^0 = \sqrt{-2H}[-2HQ^0 + (Q\cdot P)]/\alpha. \tag{72}$$

The basis for the spectrum generating algebra presented in [2] and in case 1 of [43] is composed of the 15 quantities $\mathcal{M}_{0i}, \mathcal{M}_{ij}, \mathcal{D}, \mathcal{V}_0, \mathcal{V}_i, \mathcal{E}, \mathcal{S}_i$. It is important to stress that the Poisson bracket commutation relations for the spectrum generating algebra presented in [2, 43] are the *equal time* Poisson brackets $\{C_I, C_J\}^3$.

We can easily obtain similar expressions for the case of positive energies,

$$\begin{aligned}
&\mathcal{T}_0 = \tfrac{1}{2}(\mathcal{V}_0 + \mathcal{E}) \qquad \mathcal{T}_i = \tfrac{1}{2}(\mathcal{V}_i + \mathcal{S}_i) \\
&\mathcal{M}_{0i} = |Q|P_i\cosh\beta - \sqrt{1/2H}\left[-\frac{\alpha}{|Q|}Q^i + (Q\cdot P)P_i\right]\sinh\beta \\
&\mathcal{M}_{ij} = Q^iP_j - Q^jP_i \\
&\mathcal{D} = -\sqrt{1/2H}(2H|Q| + \alpha)\sinh\beta + (Q\cdot P)\cosh\beta \\
&\mathcal{K}_0 = \frac{-4\sinh\beta}{\sqrt{-k}[\cosh\beta + (|P|^2 + 2H)|Q|/2\alpha]}\mathcal{D} + \frac{16[-\alpha^2\sinh^2\beta + 2H|P|^2|Q|^2]}{k[2\alpha\cosh\beta + (|P|^2 + 2H)|Q|]^2}\mathcal{T}_0 \\
&\mathcal{K}_i = \frac{8\sqrt{-2H}P_i}{[(2\alpha/|Q|)\cosh\beta + (|P|^2 + 2H)]}\mathcal{D} + \frac{16[-\alpha^2\sinh^2\beta + 2H|P|^2|Q|^2]}{k[2\alpha\cosh\beta + (|P|^2 + 2H)|Q|]^2}\mathcal{T}_i
\end{aligned} \tag{73}$$

where

$$\begin{aligned}
&\mathcal{V}_0 = -\alpha\sqrt{-k/2H} \qquad \mathcal{V}_i = \sqrt{-k/2H}[-(|P|^2 + 2H)Q^i + 2(Q\cdot P)P_i]/2 \\
&\mathcal{E} = -\sqrt{-k/2H}(2H|Q| + \alpha)\cosh\beta + \sqrt{-k}(Q\cdot P)\sinh\beta \\
&\mathcal{S}_i = -\sqrt{-k}|Q|P_i\sinh\beta + \sqrt{-k/2H}\left[-\frac{\alpha}{|Q|}Q^i + (Q\cdot P)P_i\right]\cosh\beta
\end{aligned} \tag{74}$$

and β is defined as

$$\beta = \sqrt{-k}\bar{x}^0 = \sqrt{2H}[-2HQ^0 + (Q\cdot P)]/\alpha. \tag{75}$$

5.2. Zero energy states

In view of equation (54), we must look elsewhere for the case of zero energy states in the Kepler problem. In order to treat the case of zero energy, we consider the null geodesic motion in Minkowski spacetime (39), i.e. $\mathcal{G} = 0$ and $k = 0$ [69]. We integrate expression (51) for τ for the case where $k = 0$ and find that it is not possible to extend this to a canonical transformation on the eight-dimensional phase space. The time transformation is used to obtain explicit expressions for the time-dependent configurational invariants for the Kepler motion and we investigate the Poisson bracket structure.

It is straightforward to show that (51) can be integrated to give the following expression for τ, and its inverse,

$$4\tau = \tfrac{1}{3}|y|^4\lambda^3 - (x\cdot y)|y|^2\lambda^2 + |y|^2|x|^2\lambda + 4c \tag{76}$$

$$\lambda = -[J(w - w^{-1}) - (x\cdot y)]/|y|^2 \tag{77}$$

where $J^2 = |x|^2|y|^2 - (x \cdot y)^2$ is the square of the angular momentum and w is defined below. The flows are related by (8) and we emphasize that τ *does not* have the status of a canonical time coordinate. However, we can still make an overall parameter change $\lambda \mapsto \tau$, as in equation (8), ensuring that the first integrals of null geodesic motion in Minkowski spacetime become configurational invariants of Kepler motion for $H = E = 0$.

Let us verify expressions (76) and (77). For the case $k = 0$ only we find that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(x \cdot y) = \hat{\mathbf{X}}^3_G(x \cdot y) = 2G. \tag{78}$$

We also have (in fact the following equation applies for *all* values of k)

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\lambda} = G(x \cdot y). \tag{79}$$

From these two equations, it follows that

$$\mathcal{F} = G^2\lambda^2 + a\lambda + b \tag{80}$$

where a and b are constants. Explicitly

$$a = G(x \cdot y) - 2G^2\lambda \tag{81}$$

We integrate to get the following transformation,

$$\tau = \tfrac{1}{3}G^2\lambda^3 + \tfrac{1}{2}a\lambda^2 + b\lambda + c \tag{82}$$

and $\mathcal{G} = 0$ implies $G = y_0^2/2 = |y|^2/2$, giving equation (76). This can be rewritten in terms of the Minkowski time $x^0 = -\lambda y_0$ and we take $y_0 = |y|$. Thus

$$4\tau = -\tfrac{1}{3}|y|(x^0)^3 - (x \cdot y)(x^0)^2 - |y|\|x|^2x^0 + 4c. \tag{83}$$

Now we wish to invert this in order to express x^0 in terms of τ. We can get rid of the squared term by making the substitution $x^0 = X^0 - (x \cdot y)/y_0$, i.e.

$$-\left(\frac{12}{y_0}\right)\tau = (X^0)^3 + mX^0 + e \tag{84}$$

where the functions m and e are given by

$$m = 3J^2/|y|^2 \qquad e = [2(x \cdot y)^3 - 3(x \cdot y)|x|^2|y|^2 - 12c|y|^2]/|y|^3. \tag{85}$$

Equation (84) can be inverted to give X^0 as a function of τ. This is done by noting that the equation

$$6\Lambda = \gamma^3 + 3\gamma$$

has the solution (section 4.1 of [77])

$$\gamma = w - w^{-1}$$

where $w^3 = 3\Lambda + s$ and $s = \sqrt{1 + 9\Lambda^2}$. In this case

$$\Lambda = -\frac{|y|^3}{6J^3}\left(e + \frac{12\tau}{|y|}\right) \qquad \gamma = |y|X^0/J.$$

Thus

$$X^0 = J(w - w^{-1})/|y|. \tag{86}$$

Finally, we can solve for x^0,

$$x^0 = [J(w - w^{-1}) - (x \cdot y)]/|y| \tag{87}$$

and substituting for λ gives equation (77).

The 15 first integrals of motion for null geodesic motion in Minkowski spacetime are (equations (42) with $k = 0$)

$$\begin{aligned}
&\mathcal{T}_0 = y_0 \qquad \mathcal{T}_i = y_i \qquad \mathcal{M}_{ij} = x^i y_j - x^j y_i \\
&\mathcal{M}_{0i} = -x^i y_0 - x^0 y_i \qquad \mathcal{D} = x^0 y_0 + x^i y_i \\
&\mathcal{K}_0 = -2x^0\mathcal{D} - (-(x^0)^2 + |x|^2)y_0 \qquad \mathcal{K}_i = 2x^i\mathcal{D} - (-(x^0)^2 + |x|^2)y_i.
\end{aligned} \tag{88}$$

These correspond to 15 configurational invariants for $H = E = 0$ in the Kepler problem. These can be expressed in Kepler coordinates $(\bar{x}^i, \bar{y}_j, \tau)$ via canonical transformations I and II,

$$\begin{aligned}
&\mathcal{T}_0 = |\bar{x}||\bar{y}|^2/2\sqrt{2} \qquad \mathcal{T}_i = [|\bar{y}|^2\bar{x}^i - 2(\bar{x}\cdot\bar{y})\bar{y}_i]/2\sqrt{2} \\
&\mathcal{M}_{ij} = \bar{x}^i\bar{y}_j - \bar{x}^j\bar{y}_i \qquad \mathcal{M}_{0i} = -|\bar{x}|\bar{y}_i + x^0[|\bar{y}|^2\bar{x}^i - 2(\bar{x}\cdot\bar{y})\bar{y}_i]/2\sqrt{2} \\
&\mathcal{D} = x^0|\bar{x}||\bar{y}|^2/2\sqrt{2} + (\bar{x}\cdot\bar{y}) \\
&\mathcal{K}_0 = -2x^0\mathcal{D} - [-(x^0)^2 + 8/|\bar{y}|^2]|\bar{x}||\bar{y}|^2/2\sqrt{2} \\
&\mathcal{K}_i = -4\sqrt{2}\mathcal{D}\bar{y}_i/|\bar{y}|^2 - [-(x^0)^2 + 8/|\bar{y}|^2][|\bar{y}|^2\bar{x}^i - 2(\bar{x}\cdot\bar{y})\bar{y}_i]/2\sqrt{2}
\end{aligned} \tag{89}$$

where x^0, given in Kepler coordinates, is as follows:

$$x^0 = 2\sqrt{2}[J(w - w^{-1}) - (\bar{x}\cdot\bar{y})]/|\bar{x}||\bar{y}|^2. \tag{90}$$

The Kepler Hamiltonian takes the form [69]

$$H = \frac{|\bar{y}|^2}{2} - \frac{\alpha}{|\bar{x}|} \tag{91}$$

where $\alpha = \sqrt{4G}$. Now, since X^0 was defined such that $y_0X^0 = \mathcal{D}$, it is a time-dependent configurational invariant. Since J is a first integral and $|y|$ is a configurational invariant, from (86) it follows that Λ, γ and

$$\pi = (|y|e + 12\tau) \tag{92}$$

are also time-dependent configurational invariants. The latter is a convenient quantity to have since it is linear in the time parameter τ.

We make an important point regarding the nature of the configurational invariants (89) and (92): since they satisfy (19), this set does not necessarily form a Lie algebra under the Poisson bracket operation, see section 2.2. However, as also noted in section 2.2, the subset of first integrals will form a Lie algebra, in this case the set of six time-independent first integrals $\mathcal{T}_i$ and $\mathcal{M}_{ij}$ form the six-dimensional *iso*(3) algebra

$$\{\mathcal{M}_i, \mathcal{M}_j\} = -\epsilon_{ij}^k\mathcal{M}_k \qquad \{\mathcal{M}_i, \mathcal{T}_j\} = -\epsilon_{ij}^k\mathcal{T}_k \qquad \{\mathcal{T}_i, \mathcal{T}_j\} = 0 \tag{93}$$

the *iso*(3) structure is guaranteed by the canonical nature of the transformations I and II (of course, this is unaffected by the subsequent non-canonical time-parameter transformation). Can the Lie algebra (93) be increased by introducing a single time-dependent configurational invariant? (Introducing two or more would introduce, via the Poisson bracket operation, quantities that are not configurational invariants.) We are free to choose any of the time-dependent configurational invariants above but we choose to consider the quantity which is linear in the Kepler time τ. Introducing $\pi' = 2\pi|y| \equiv \pi|\bar{x}||\bar{y}|^2/\sqrt{2}$ we have the following Poisson brackets,

$$\{\mathcal{M}_i, \pi'\} = 0 \qquad \{\mathcal{T}_i, \pi'\} = -12J^2\mathcal{T}_i + H\eta_i \tag{94}$$

where $\eta_i = (12|\bar{y}|^2\tau - 2(\bar{x}\cdot\bar{y}))[(\bar{x}\cdot\bar{y})\bar{x}^i - |\bar{x}|^2\bar{y}_i]$. Now, since the latter Poisson bracket produces a time-dependent configurational invariant, which is not a linear combination of the $\mathcal{T}_i$, $\mathcal{M}_{ij}$ or π', the bracket of this quantity and π' will introduce quantities which are not configurational invariants and so the set of configurational invariants (including the set of time-independent first integrals) will not form a Lie algebra. Further, it seems that the square of the angular momentum J^2 cannot be scaled out to give a Lie algebra structure $iso(3) \oplus_s d$ even on the $H = E = 0$ hypersurfaces. It turns out that there is no closed Lie algebra structure for any of the time-dependent configurational invariants above. However, by construction, we should not have expected to obtain a spectrum generating algebra allowing transitions between different Kepler energies H since the time-dependent quantities in (89) and (92) are only constants of motion for $H = E = 0$.

Thus we can only conclude that the energy eigenvalue $E = 0$ must be excluded from the domain of the standard $so(4, 2)$ spectrum generating algebra for the Kepler problem. However, the six time-independent first integrals do form a six-dimensional $iso(3)$ symmetry algebra (93).

5.3. TIRs of the spectrum generating algebras

Quantities (70) and (73) with $Q^0 = 0$ form TIRs of the spectrum generating algebras for the classical Kepler problem for non-zero values of H only. For $H < 0$ we have a convenient basis

$$
\begin{aligned}
\mathcal{M}_{0i} &= |Q| P_i \cos \Xi - \sqrt{-1/2H}\left[-\frac{\alpha}{|Q|} Q^i + (Q \cdot P) P_i\right] \sin \Xi \\
\mathcal{M}_{ij} &= Q^i P_j - Q^j P_i \\
\mathcal{D} &= -\sqrt{-1/2H}(2H|Q| + \alpha) \sin \Xi + (Q \cdot P) \cos \Xi \\
\mathcal{V}_0 &= -\alpha\sqrt{-k/2H} \qquad \mathcal{V}_i = \sqrt{-k/2H}[-(|P|^2 + 2H) Q^i + 2(Q \cdot P) P_i]/2 \\
\mathcal{E} &= -\sqrt{-k/2H}(2H|Q| + \alpha) \cos \Xi - \sqrt{k}(Q \cdot P) \sin \Xi \\
\mathcal{S}_i &= \sqrt{k}|Q| P_i \sin \Xi + \sqrt{-k/2H}\left[-\frac{\alpha}{|Q|} Q^i + (Q \cdot P) P_i\right] \cos \Xi
\end{aligned}
\tag{95}
$$

and for $H < 0$ the quantity Ξ is given by

$$
\Xi = \sqrt{-2H}(Q \cdot P)/\alpha. \tag{96}
$$

We can easily obtain similar expressions for the case $H > 0$,

$$
\begin{aligned}
\mathcal{M}_{0i} &= |Q| P_i \cosh \Xi - \sqrt{1/2H}\left[-\frac{\alpha}{|Q|} Q^i + (Q \cdot P) P_i\right] \sinh \Xi \\
\mathcal{M}_{ij} &= Q^i P_j - Q^j P_i \\
\mathcal{D} &= -\sqrt{1/2H}(2H|Q| + \alpha) \sinh \Xi + (Q \cdot P) \cosh \Xi \\
\mathcal{V}_0 &= -\alpha\sqrt{-k/2H} \qquad \mathcal{V}_i = \sqrt{-k/2H}[-(|P|^2 + 2H) Q^i + 2(Q \cdot P) P_i]/2 \\
\mathcal{E} &= -\sqrt{-k/2H}(2H|Q| + \alpha) \cosh \Xi + \sqrt{-k}(Q \cdot P) \sinh \Xi \\
\mathcal{S}_i &= -\sqrt{-k}|Q| P_i \sinh \Xi + \sqrt{-k/2H}\left[-\frac{\alpha}{|Q|} Q^i + (Q \cdot P) P_i\right] \cosh \Xi
\end{aligned}
\tag{97}
$$

and for $H > 0$ the quantity Ξ is given by

$$\Xi = \sqrt{2H}(Q \cdot P)/\alpha. \tag{98}$$

We observe that in both spectrum generating algebras (95) and (97) the quantity $\mathcal{V}_0$ is a function of the Hamiltonian H, i.e. the Hamiltonain does indeed map the algebra into itself, as we would expect. Now, one can consider the eight non-compact generators $\mathcal{M}_{0i}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{S}_i$, writing them collectively as J_A, $A = 1, \ldots, 8$. Then a canonical transformation of the form [78]

$$J'_a = \exp\widetilde{(n \cdot J)} J_a = R_a^b(n) J_b \tag{99}$$

gives a new canonical basis for negative energies [43, 55]

$$\begin{aligned} &\mathcal{A}_i = [|P|^2 Q^i - 2(Q \cdot P)P_i]/2\sqrt{2} \qquad \mathcal{M}_i = \epsilon_{ij}^k Q^j P_k \\ &\mathcal{N} = (Q \cdot P) \qquad \mathcal{B}_i = -2\sqrt{2}Q^i \qquad \mathcal{I}_i = -\sqrt{2}|Q|P^i \\ &\sqrt{G_\pm} = (|P|^2 \pm 2k)|Q|/4 \end{aligned} \tag{100}$$

the notation is chosen in accordance with the appendix. One can carry out the same procedure for positive energies (97) and we obtain the same basis (100). Thus, (100) is an alternative realization of the TIR of the spectrum generating algebra for the classical Kepler problem for non-zero energies.

The quantities (95) and (97) constitute the TIR of the spectrum generating algebra $so(4, 2)$ for negative and positive energies respectively in the classical Kepler problem and those which do not commute with the Hamiltonian H correspond to transitions between the different energy states with energy $H < 0$ and $H > 0$ respectively.

From the results obtained in section 5.2 we must conclude that the operators (95) and (97), or equivalently (100), *do not* constitute a TIR of the spectrum generating algebra for the case of zero energy in the classical Kepler problem. We emphasize the following: the quantities (50) constitute the TIR of the spectrum generating algebra $so(4, 2)$ for the geodesic motion problem for $G > 0$ and those which do not commute with the Hamiltonian G correspond to transitions between the different energy states with energy $G > 0$. We note that we can use the canonical transformations I and II to obtain from the quantities (50) directly the spectrum generating algebras for the non-zero energies in the classical Kepler problem (100). We remark that placing $k = 0$ in (50) *does not* give a spectrum generating algebra for $E = H = 0$ in the Kepler problem. One does have a representation of the $so(4, 2)$ algebra but it has no dynamical role (except for the $iso(3)$ subgroup).

In the appendix, we give an alternative geometrical interpretation of the TIR of the spectrum generating algebra (50).

6. Conclusions

We have shown that the equations of null geodesic motion in Einstein static spacetimes of arbitrary curvature are directly related to those of the classical Kepler problem and that the value of the energy in the latter is proportional to (minus) the curvature parameter in the former. This work extended the formalism in [69] to derive the time-dependent first integrals of motion for non-zero energies in the classical Kepler problem, and time-dependent configurational invariants for the case of zero energy. We summarize the results in the following two diagrams. We have the following mappings for *non-zero energy states*:

$$
\begin{array}{rl}
(T^*M, \tilde{\Omega}, \mathcal{G} = 0) & (x^0, y_0, x^i, y_j; \lambda) \\
\Big\downarrow & \text{canonical transformations I, II and III} \\
(T^*M, \tilde{\Omega}, \mathcal{G} = 0) & (q^0, p_0, q^i, p_j; \lambda) \\
\Big\downarrow & \text{canonical transformation IV } (k \neq 0) \\
(T^*M, \tilde{\Omega}, \mathcal{H} = 0) & (Q^0, P_0, Q^i, P_j; \Upsilon) \\
\Big\downarrow & \text{reduction } (H \neq 0) \\
(T^*\Gamma, \tilde{\omega}, H) & (Q^i, P_j; Q^0).
\end{array}
$$

The result was achieved by exploiting the parallel Hamiltonian vector fields (equation (62))

$$\hat{\mathbf{X}}^4_{\mathcal{G}} = \frac{\mathrm{d}\Upsilon}{\mathrm{d}\lambda} \hat{\mathbf{X}}^4_{\mathcal{H}}$$

for non-zero energies $H = E \neq 0$. The time-independent and time-dependent first integrals form a Lie algebra in the case of non-zero energies, that algebra being $so(4, 2)$. This was guaranteed by the canonical nature of the transformations and reduction procedure. To summarize, we have shown that it is the existence of a TDR of the $so(4, 2)$ spectrum generating algebra for null geodesic motion in the Einstein static spacetimes (originating from the $so(4, 2)$ algebra of first integrals) which determines the corresponding spectrum generating algebra structure in the classical Kepler problem.

We have the following mappings for *zero energy states*:

$$
\begin{array}{rl}
(T^*M, \tilde{\Omega}, \mathcal{G} = 0) & (x^0, y_0, x^i, y_j; \lambda) \\
\Big\downarrow & \text{canonical transformations I and II} \\
(T^*M, \tilde{\Omega}, \mathcal{G} = 0) & (\bar{x}^0, \bar{y}_0, \bar{x}^i, \bar{y}_j; \lambda) \\
\Big\downarrow & \text{non-canonical transformation (90)} \\
(T^*M, \tilde{\Omega}, \mathcal{H} = 0) & (\bar{x}^0, \bar{y}_0, \bar{x}^i, \bar{y}_j; \tau) \\
\Big\downarrow & \text{reduction } (H = 0) \\
(T^*\Gamma, \tilde{\omega}, H) & (\bar{x}^i, \bar{y}_j; \tau).
\end{array}
$$

The result was achieved by exploiting the parameter change (equation (8)) which ensured the correspondence of configurational invariants for both systems. For the case of zero energy, only the time-independent configurational invariants form a Lie algebra, that Lie algebra being *iso*(3). We conclude that the spectrum generating algebras appropriate for non-zero energies do not extend to the case of zero energy. The *so*(4, 2) generators will generate canonical transformations, of these however, only the elements of the *iso*(3) invariance algebra will generate canonical transformations which map Kepler orbits to Kepler orbits and further, these Kepler orbits will all have the same energy $E = 0$. This is supported by the analysis of Cariñena *et al* [68] where they are forced to construct two separate *SO*(3, 2) dynamical groups for negative and positive energies. This is also apparent from the quantum mechanical analogue, since the operators which allow transitions between negative energy states (similarly for positive energy states) do not provide transitions to zero energy states [46, 48, 49]. Whether there are alternative methods available to construct similar, or alternative, Lie algebras extending to the case of zero energy is unknown. It is important to emphasize that one can always construct an *so*(4, 2) algebra with structure constants independent of $E = H$, i.e. those given by the commutation relations (41). It follows that if we choose a specific non-zero value of energy E then it is *not* necessary to scale out this value to obtain the spectrum generating algebras. For $E = 0$ one still has an *so*(4, 2) algebra structure but it seems that only the *iso*(3) subalgebra has a physical significance.

The case $E = 0$ is the energy for which (3.15) in [54] is invalid. This is due to the fact that the corresponding time coordinate is, by definition, a constant of motion in that case. For $E = 0$ one must instead use the non-canonical time-parameter transformation (76) presented above. Cordani [54] defines Fock and Bacry–Györgyi variables for $E = 0$ based on the assumption that (3.15) in [54] applies, however one must be careful in that these do not have the same meaning as for the case $E \neq 0$, i.e. they do not provide transitions to non-zero energy states. These variables are indeed the generators of the spectrum generating group (50) for geodesic motion on Γ for arbitrary values of k, however, placing $k = 0$ in (50) *does not* give the spectrum generating algebra including $H = E = 0$ in the Kepler problem. On placing $k = 0$ one still has a representation of an *so*(4, 2) algebra but, as we have mentioned, the elements of the Lie algebra are not transitive on the space of Kepler orbits of arbitrary energy. Further, the quantities X and $\bar{X}$ in [79] are undefined for $H = 0$ and so, contrary to their claims, their remarks do not apply to zero energy states.

We refer the reader to Sudarshan and Mukunda [39], Cariñena *et al* [68] and McAnally and Bracken [79] for an account of the action of the spectrum generating group in the classical Kepler problem.

A *dynamical symmetry* of a system is the most general type of transformation mapping solutions into solutions [71]. One can associate first integrals with certain subclasses of such transformations, and the spectrum generating algebras constitute such a set. Any element of the invariance group of a system which arises from the regularities in the phase space of Hamiltonian dynamics and which is not immediately apparent from an inspection of the geometrical symmetries of the potential itself is usually referred to as a *hidden symmetry* [80]. Thus, in the case of the Kepler problem the subgroup of symmetry transformations generated by the components of the Laplace–Runge–Lenz vector are hidden symmetries. The above analysis has shown that the spectrum generating algebra arises as a result of the existence of conformal symmetries of Minkowski spacetime and certain spacetimes conformally related to them. In particular, the hidden symmetries arise as a result of an isometry subgroup of the relevant spacetime.

We hope that we have clarified the relationship between the Lie group *SO*(4, 2) as the Lie group of conformal symmetries of Minkowski spacetime and as the spectrum generating group of the classical Kepler problem.

Table 2. The conformal scalars for the CKV of (Γ, g).

Vector field	Conformal scalar	Type
$\mathbf{A}_i$	$\phi = -kK_+^{-1}x^i/2$	CKV
$\mathbf{M}_i$	$\phi = 0$	KV
$\mathbf{N}$	$\phi = K_- K_+^{-1}$	CKV
$\mathbf{B}_i$	$\phi = 2K_+^{-1}x^i$	CKV

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Appendix A. Conformal symmetries of Γ

A1. Geometry

The three-dimensional manifold Γ is conformally Euclidean and so the CKVs for Γ are just the CKVs for the flat three-dimensional Euclidean space $\mathbf{E}^3$, see section 3. A basis for the CKV of $\mathbf{E}^3$ is as follows,

$$\mathbf{A}_i = \frac{\partial}{\partial x^i} \qquad \mathbf{M}_i = \epsilon^k_{ij} x^j \frac{\partial}{\partial x^k} \qquad \mathbf{N} = x^i \frac{\partial}{\partial x^i} \qquad \mathbf{B}_i = 2x^i \mathbf{N} - |\mathbf{x}|^2 \frac{\partial}{\partial x^i} \tag{101}$$

where $\mathbf{M}_i = \epsilon^k_{ij} \mathbf{M}^j_k$ with $\mathbf{M}^j_k$ defined as in (42). The functions ϕ for these vector fields in Γ are shown in table 2. We now present the commutation relations for the basis of CKV (101):

$$\begin{array}{lll} [\mathbf{M}_i, \mathbf{M}_j] = -\epsilon^k_{ij}\mathbf{M}_k & [\mathbf{M}_i, \mathbf{A}_j] = -\epsilon^k_{ij}\mathbf{A}_k & [\mathbf{A}_i, \mathbf{A}_j] = 0 \\ [\mathbf{B}_i, \mathbf{B}_j] = 0 & [\mathbf{B}_i, \mathbf{M}_j] = -\epsilon^k_{ij}\mathbf{B}_k & [\mathbf{A}_i, \mathbf{B}_j] = 2\delta_{ij}\mathbf{N} - 2\epsilon^k_{ij}\mathbf{M}_k \\ [\mathbf{B}_i, \mathbf{N}] = -\mathbf{B}_i & [\mathbf{M}_i, \mathbf{N}] = 0 & [\mathbf{A}_i, \mathbf{N}] = \mathbf{A}_i. \end{array} \tag{102}$$

The Lie algebra (102) is isomorphic to the Lie algebra $so(4, 1)$, see for example [76]. As we would expect, this Lie algebra is independent of the curvature k. From equation (24) we can see that the Poisson bracket Lie algebra of quantities $\mathbf{X}_I \cdot \mathbf{y}$ will have the same structure constants as the Lie algebra of CKV (102) and so this Poisson bracket Lie algebra will also be isomorphic to $so(4, 1)$. We shall now present the associated quantities $\mathbf{X}_I \cdot \mathbf{y}$. We shall write $\mathcal{A}_i = \mathbf{A}_i \cdot \mathbf{y}$ etc,

$$\mathcal{A}_i = y_i \qquad \mathcal{M}_i = \epsilon^k_{ij} x^j y_k \qquad \mathcal{N} = (x \cdot y) \qquad \mathcal{B}_i = 2x^i (x \cdot y) - |\mathbf{x}|^2 y_i. \tag{103}$$

These are the components of the the vector fields $\mathbf{A}_i$, $\mathbf{M}_i$, $\mathbf{N}$ and $\mathbf{B}_i$ along the geodesic tangent vector $\mathbf{y}$. Only a subset of the quantities (103) will be first integrals of motion. At this point, we note that the $\mathbf{M}_i$ are always KVs for Γ and from table 2 we can see that the following linear combination of the CKV will always be a KV

$$\mathbf{P}_i = \mathbf{A}_i + \frac{k}{4}\mathbf{B}_i. \tag{104}$$

Thus it is always possible to find a basis which includes a six-dimensional KV subalgebra which consists of $\mathbf{M}_i$ and $\mathbf{P}_i$. From these KVs we can construct the corresponding six first integrals $\mathcal{M}_i$ and $\mathcal{P}_i = \mathbf{P}_i \cdot \mathbf{y}$ and we note that

$$\mathcal{P}_i = \mathcal{A}_i + \frac{k}{4}\mathcal{B}_i. \tag{105}$$

A2. Dynamics on Γ

Let us define $G_{\pm} = K_{\pm}^2|\mathbf{y}|^2/2$. Then consider geodesic motion on Γ as given by the Hamiltonian function G_+. The first integrals $\mathcal{M}_i$ and $\mathcal{P}_i$ satisfy

$$\hat{\mathbf{X}}_{G_+}(\mathcal{M}_i) = 0 \qquad \hat{\mathbf{X}}_{G_+}(\mathcal{P}_i) = 0 \tag{106}$$

that is, the components of the KVs $\mathbf{R}_i$ and $\mathbf{P}_i$ are constant along the geodesic flow. Thus, it is natural to investigate also the variation of the components of the CKVs, given by (103), along the geodesic flow. We find that for each of $\mathcal{A}_i, \mathcal{M}_i, \mathcal{N}$ and $\mathcal{B}_i$, equation (33) and table 2 give the following,

$$\begin{aligned} &\hat{\mathbf{X}}_{G_+}(\mathcal{A}_i) = -k\sqrt{G_+}\mathcal{I}_i/2 && \hat{\mathbf{X}}_{G_+}(\mathcal{M}_i) = 0 \\ &\hat{\mathbf{X}}_{G_+}(\mathcal{N}) = 2\sqrt{G_+}\sqrt{G_-} && \hat{\mathbf{X}}_{G_+}(\mathcal{B}_i) = \sqrt{4G_+}\mathcal{I}_i \end{aligned} \tag{107}$$

where

$$\mathcal{I}_i = \sqrt{2}|\mathbf{y}|x^i. \tag{108}$$

This leads to the Poisson brackets

$$\begin{aligned} &\{\sqrt{G_+}, \mathcal{A}_i\} = -k\mathcal{I}_i/4 && \{\sqrt{G_+}, \mathcal{M}_i\} = 0 \\ &\{\sqrt{G_+}, \mathcal{N}\} = -\sqrt{G_-} && \{\sqrt{G_+}, \mathcal{B}_i\} = \mathcal{I}_i. \end{aligned} \tag{109}$$

The quantities (109) are the rates of change of the quantities $\mathbf{X}_I \cdot \mathbf{y}$, given by (103), along the phase flow generated by the Hamiltonian function $\sqrt{G_+}$. The complete Lie algebra structure is as follows:

$$\begin{aligned} &\{\sqrt{G_+}, \mathcal{A}_i\} = -k\mathcal{I}_i/4 && \{\sqrt{G_+}, \mathcal{M}_i\} = 0 \\ &\{\sqrt{G_+}, \mathcal{N}\} = -\sqrt{G_-} && \{\sqrt{G_+}, \mathcal{B}_i\} = \mathcal{I}_i \\ &\{\sqrt{G_-}, \mathcal{A}_i\} = k\mathcal{I}_i/4 && \{\sqrt{G_-}, \mathcal{M}_i\} = 0 \\ &\{\sqrt{G_-}, \mathcal{N}\} = -\sqrt{G_+} && \{\sqrt{G_-}, \mathcal{B}_i\} = \mathcal{I}_i \\ &\{\mathcal{I}_i, \mathcal{I}_j\} = 2\mathcal{M}_{ij} && \{\sqrt{G_+}, \sqrt{G_-}\} = -k\mathcal{N}/2 \\ &\{\mathcal{I}_i, \mathcal{A}_j\} = -\delta_{ij}(\sqrt{G_+} + \sqrt{G_-}) && \{\mathcal{I}_i, \mathcal{N}\} = 0 \\ &\{\mathcal{I}_i, \mathcal{B}_j\} = -4\delta_{ij}(\sqrt{G_+} - \sqrt{G_-})/k. \end{aligned} \tag{110}$$

The 15 quantities $\mathcal{A}_i, \mathcal{M}_i, \mathcal{N}, \mathcal{B}_i, \mathcal{I}_i, \sqrt{G_-}, \sqrt{G_+}$ constitute a TIR of the spectrum generating algebra $so(4, 2)$ for non-zero values of the curvature k.

A3. TIR of the spectrum generating algebra for the classical Kepler problem

Under the canonical transformations I and II the members of the algebra become

$$\begin{aligned} &\mathcal{A}_i = [|\bar{y}|^2\bar{x}^i - 2(\bar{x}\cdot\bar{y})\bar{y}_i]/2\sqrt{2} \qquad \mathcal{M}_i = \epsilon^k_{ij}\bar{x}^j\bar{y}_k \\ &\mathcal{N} = (\bar{x}\cdot\bar{y}) \qquad \mathcal{B}_i = -2\sqrt{2}\bar{x}^i \qquad \mathcal{I}_i = -\sqrt{2}|\bar{x}|\bar{y}^i \\ &\sqrt{G_{\pm}} = (|\bar{y}|^2 \pm 2k)|\bar{x}|/4. \end{aligned} \tag{111}$$

Since the transformations are canonical these quantities still constitute a basis for the Lie algebra $so(4, 1)$. The quantities $\sqrt{G_+}, \sqrt{G_-}$ and $\mathcal{N}$ give the $so(2, 1)$ energy spectrum generating algebra for non-zero values of k [16, 48]. Thus the quantities $\mathcal{A}_i, \mathcal{M}_i, \mathcal{N}$ and $\mathcal{B}_i$ can be thought of as having a geometrical origin in that they arise as a result of the CKV of $\mathbf{E}^3$. We note that the quantities $\mathcal{P}_i$ in equation (105) are the components of the Laplace–Runge–Lenz vector.

The above quantities are immediately recognizable (up to constant multiples) as the *so*(4, 2) generators for positive and negative energy states corresponding to those generators given by [16, 41–43, 47, 48, 54, 56, 63].

References

[1] Goldstein H 1980 *Classical Mechanics* (Reading, MA: Addison-Wesley)
[2] Dothan Y 1970 *Phys. Rev.* D **2** 2944
[3] Hermann R 1972 *J. Math. Phys.* **13** 833
[4] Hermann R 1972 *J. Math. Phys.* **13** 838
[5] Andrié M 1976 *J. Math. Phys.* **17** 394
[6] Barut A O 1964 *Symmetry Principles at High Energy, Coral Gables Conference* ed B Kursunogla and A Perlmutter (San Francisco, CA: Freeman) p 81
[7] Barut A O and Böhm A 1965 *Phys. Rev.* **139** B1107
[8] Dothan Y, Gell-Mann M and Ne'eman Y 1965 *Phys. Lett.* **17** 148
[9] Mukunda N, O'Raifeartaigh L and Sudarshan E C G 1965 *Phys. Rev. Lett.* **15** 1041
[10] Bacry H, Ruegg H and Souriau J M 1966 *Commun. Math. Phys.* **3** 323
[11] Mukunda N 1967 *Phys. Rev.* **155** 1383
[12] Böhm A 1967 *Lectures in Theoretical Physics* vol 9-B (London: Gordon and Breach) p 327
[13] Bacry H 1967 *Lectures in Theoretical Physics* vol 9-A (London: Gordon and Breach) p 79
[14] Vitale B 1969 *Lectures in Theoretical Physics* vol 11 (London: Gordon and Breach) p 405
[15] Lipkin H J 1968 *Symmetry Principles at High Energy, 5th Coral Gables Conference* ed A Perlmutter *et al* (New York: Benjamin) p 261
[16] Mariwalla K 1975 *Phys. Rep.* **20** 287
[17] Böhm A, Ne'eman Y and Barut A O 1988 *Dynamical Groups and Spectrum Generating Algebras* vol 1 and 2 (Singapore: World Scientific)
[18] Castaños O, Frank A and Lopez-Peña R 1990 *J. Phys. A: Math. Gen.* **23** 5141
[19] Ermanno G J 1710 *G. Lett. Ital.* **2** 447
[20] Bernoulli J 1710 *Hist. Acad. R. Sci: Mém. Math. Phys.* 521
[21] Hermann J 1710 *Hist. Acad. R. Sci: Mém. Math. Phys.* 519
[22] Laplace P S 1827 *A Treatise of Celestial Mechanics* (Dublin)
[23] Runge C 1919 *Vektoranalysis* vol 1 (Hirzel, Leipzig)
[24] Lenz W 1924 *Z. Phys.* **24** 197
[25] Pauli W 1926 *Z. Phys.* **36** 336
[26] Hulthén L 1933 *Z. Phys.* **86** 21
[27] Fock V 1935 *Z. Phys.* **98** 145
[28] Bargmann V 1936 *Z. Phys.* **99** 576
[29] Moser J 1970 *Commun. Pure Appl. Math.* **23** 609
[30] Belbruno E A 1977 *Celest. Mech.* **15** 467
[31] Osipov Y S 1977 *Celest. Mech.* **16** 191
[32] Souriau J M 1974 *Symp. Math.* **14** 343
[33] Quan P M 1980 *C. R. Acad. Sci., Paris* **291** 219
[34] Quan P M 1980 *C. R. Acad. Sci., Paris* **291** 299
[35] Milnor J 1983 *Am. Math. Mon.* **90** 353
[36] Bacry H 1966 *Nuovo Cimento* A **41** 222
[37] Bander M and Itzykson C 1966 *Rev. Mod. Phys.* **38** 330
[38] Bander M and Itzykson C 1966 *Rev. Mod. Phys.* **38** 346
[39] Sudarshan E C G and Mukunda N 1966 *Lectures in Theoretical Physics* vol 8-B (Boulder, CO: University of Colorado) p 407
[40] Györgyi G 1968 *Nuovo Cimento* A **53** 717
[41] Györgyi G 1969 *Nuovo Cimento* A **62** 449
[42] Barut A O and Bornzin G L 1971 *J. Math. Phys.* **12** 841
[43] Tripathy K C, Gupta R and Anand J D 1975 *J. Math. Phys.* **16** 1139
[44] Malkin I A and Man'ko V I 1965 *JETP Lett.* **2** 146
[45] Kleinert H 1968 *Lectures in Theoretical Physics* vol 10-B (London: Gordon and Breach) p 427
[46] Englefield M J 1972 *Group Theory and the Coulomb Problem* (New York: Wiley-Interscience)

[47] Barut A O 1972 *Dynamical Groups and Generalised Symmetries in Quantum Theory,* (Christchurch: University of Canterbury Press)
[48] Wybourne B G 1974 *Classical Groups for Physicists* (New York: Wiley-Interscience)
[49] Barut A O 1973 *SIAM J. Appl. Math.* **25** 247
[50] Lindblad G and Nagel B 1970 *Ann. Inst. H. Poincaré* **12** 27
[51] Hughston L P and Hurd T R 1983 *Phys. Rep.* **100** 273
[52] Hurd T R 1985 *Proc. R. Soc.* A **397** 233
[53] Penrose R and Rindler W 1986 *Spinors and Space-time Volume 2: Spinor and Twistor Methods in Space-time Geometry* (Cambridge: Cambridge University Press)
[54] Cordani B 1986 *Commun. Math. Phys.* **103** 403
[55] Cordani B *Kepler Problem and Celestial Mechanics* (Basle: Birkhäuser) at press
[56] Barut A O and Bornzin G L 1974 *J. Math. Phys.* **15** 1000
[57] Onofri E and Pauri M 1972 *J. Math. Phys.* **13** 533
[58] Onofri E 1976 *J. Math. Phys.* **17** 401
[59] Akyildiz Y 1980 *J. Math. Phys.* **21** 665
[60] Cordani B 1988 *Commun. Math. Phys.* **113** 649
[61] Cordani B 1989 *J. Phys. A: Math. Gen.* **22** 2695
[62] Guillemin V and Sternberg S 1990 *Variations on a Theme by Kepler* vol 42 (Providence, RI: American Mathematical Society Colloquium Publications)
[63] Baumgarte J 1979 *J. Appl. Math. Mech.* **59** 177
[64] Stiefel E L and Scheifele G 1971 *Linear and Regular Celestial Mechanics* (Berlin: Springer)
[65] Kummer M 1982 *Commun. Math. Phys.* **84** 133
[66] Iwai T 1981 *J. Math. Phys.* **22** 1633
[67] Mladenov I M 1989 *Ann. Inst. H. Poincaré* **50** 219
[68] Cariñena J F, Lopez C, Del Olmo M A and Santander M 1991 *Celest. Mech.* **52** 307
[69] Keane A J, Barrett R K and Simmons J F L 2000 *J. Math. Phys.* **41** 8108
[70] Keane A J and Barrett R K 2000 *Class. Quantum Grav.* **17** 201
[71] Stephani H 1989 *Differential Equations: Their Solution Using Symmetries* (Cambridge: Cambridge University Press)
[72] Hall L S 1983 *Physica* D **8** 90
[73] Sarlet W, Leach P G L and Cantrijn F 1985 *Physica* D **17** 87
[74] Hall L S 1987 *Physica* D **24** 406
[75] Landau L D and Lifshitz E M 1960 *Mechanics* (Oxford: Pergamon)
[76] Gilmore R 1974 *Lie Groups, Lie Algebras and Some of Their Applications* (New York: Wiley-Interscience)
[77] Neutsch W and Scherer K 1992 *Celestial Mechanics* (B I Wissenschaftsverlag, Mannheim/Leipzig/Wein/Zurich)
[78] Chand P, Mehta C L, Mukunda N and Sudarshan E C G 1967 *J. Math. Phys.* **8** 2048
[79] McAnally D S and Bracken A J 1988 *Phys. Rev.* A **37** 2304
[80] Cisneros A and McIntosh H V 1970 *J. Math. Phys.* **11** 870